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GENERATION OF GREY PATTERNS USING AN IMPROVED GENETIC-EVOLUTIONARY ALGORITHM: SOME NEW RESULTS*

Alfonsas Misevičius

Kaunas University of Technology, Department of Multimedia Engineering,
Studentų St. 50–400a/416a, LT–51368 Kaunas, Lithuania
e-mail: alfonsas.misevicius@ktu.lt

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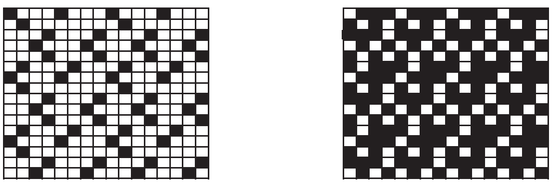
Abstract. Genetic and evolutionary algorithms have achieved impressive success in solving various optimization problems. In this work, an improved genetic-evolutionary algorithm (IGEA) for the grey pattern problem (GPP) is discussed. The main improvements are due to the specific recombination operator and the modified tabu search (intra-evolutionary) procedure as a post-recombination algorithm, which is based on the intensification and diversification methodology. The effectiveness of IGEA is corroborated by the fact that all the GPP instances tested are solved to pseudo-optimality at very small computational effort. The graphical illustrations of the grey patterns are presented.

Keywords: combinatorial optimization; heuristics; genetic-evolutionary algorithms; grey pattern problem.

Introduction

The grey pattern problem (GPP) [29] deals with a rectangle (grid) of dimensions $n_1 \times n_2$ containing $n = n_1 \times n_2$ points (m black points and $n - m$ white points). By putting together many of these rectangles, one gets a grey pattern (frame) of density m/n (see Figure 1). The goal is to get the most excellent grey pattern, that is, the points have to be spread on the rectangles as smoothly as possible.



(a) **(b)**

Figure 1. Examples of grey frames: m/n = 58/256 (a), m/n = 198/256 (b)

Formally, the GPP can be stated as follows. Let two matrices $\boldsymbol{A} = (a_{ij})_{n\times n}$ and $\boldsymbol{B} = (b_{kl})_{n\times n}$ and the set Π of all possible permutations of the integers from 1 to n be given. The objective is to find a permutation $\pi = (\pi(1), \pi(2), ..., \pi(n)) \in \Pi$ that minimizes

$$z(\pi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\pi(i)\pi(j)} , \qquad (1)$$

where the matrix $(a_{ij})_{n\times n}$ is defined as $a_{ij} = 1$ for $i, j = 1, 2, ..., m$ and $a_{ij} = 0$ otherwise; the matrix $(b_{kl})_{n\times n}$ is defined by the distances between every two of n points. More precisely, $b_{kl} = b_{n_2(r-1)+s\ n_2(t-1)+u} = f_{rstu}$, where

$$f_{rstu} = \max_{v,w\in\{-1,0,1\}} \frac{1}{(r-t+n_1 v)^2 + (s-u+n_2 w)^2} ,$$
$$r, t = 1, ..., n_1,\ s, u = 1, ..., n_2. \qquad (2)$$

f_{rstu} may be thought of as an electrical repulsion force between two electrons (to be put on the grid points) i and j ($i, j = 1, ..., n$) located in the positions $k = \pi(i)$ and $l = \pi(j)$ with the coordinates (r, s) and (t, u). The ith ($i \le m$) element of the permutation π, $\pi(i) = n_2(r-1) + s$, gives the location in the rectangle where a black point has to be placed in. The coordinates of the black point are derived according to the formulas: $r = \lfloor(\pi(i)-1)/n_2\rfloor + 1$, $s = ((\pi(i)-1) \bmod n_2) + 1$, where $\pi(i)$ denotes the location of the black point, $i = 1, 2, ..., m$ [29].

The grey pattern problem is a special case of the quadratic assignment problem (QAP) [4, 11], which is known to be NP-hard. Recently, genetic and evolutionary algorithms (GAs, EAs) are among the most advanced heuristic approaches for these problems [5, 15, 17, 18, 24, 29]. GAs and EAs belong to a class of population-based heuristics. The following are the main phases of the genetic-evolutionary algorithms. Usually, a pair of individuals (solutions) of a population is

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selected to be parents (predecessors). A new solution (i.e. offspring) is created by recombining (merging) the parents. In addition, some individuals may undergo mutations. Afterward, a replacement (reproduction) scheme is applied to the previous generation and the offspring to determine which individuals survive to form the next generation. Over many generations, less fit individuals tend to die-off, while better individuals tend to predominate. The process continues until a termination criterion is met. For a more thorough discussion on the principles of GAs and EAs, the reader is addressed to [2, 8, 9, 12].

This work is organized as follows. In Section 1, the improved genetic-evolutionary algorithm (IGEA) framework is discussed. The details of IGEA for the grey pattern problem are described in Section 2. In Section 3, we present the results of generation of grey patterns using two versions of the proposed algorithm. Section 4 completes the work with conclusions.

1. The improved genetic-evolutionary algorithm framework

The original concepts of GAs and EAs were introduced by Holland [10], Rechenberg [25] and Schwefel [26] in the nineteen seventies. Since that time, the structure of GAs and EAs has experienced various transformations. The state-of-the-art genetic-evolutionary algorithms are in fact hybrid algorithms, which incorporate additional heuristic procedures [21, 22, 23]. However, applying hybridized algorithms does not necessarily imply that good solutions are achieved at reasonable time [17]. Indeed, hybrids often make use of refined heuristics (like simulated annealing, tabu search) which are quite time-expensive. This may become in real earnest a serious drawback, especially as we are willing to construct algorithms that are competitive with other intelligent optimization techniques. Under these circumstances, it is important to add additional enhancements to the genetic-evolutionary algorithms (see also [17]).

▪ The enhanced hybrid genetic-evolutionary algorithms should incorporate fast local improvement procedures. Only short time behaviour matters as long as we are speaking of the fast procedures in the context of hybrid GA/EAs.

▪ The compactness of the population is of great importance. In the presence of powerful local improvement procedures, the large populations are not necessary: the small population size is fully compensated by the improvement heuristic.

▪ A high degree of the diversity within the population must be maintained. This is especially true for tiny populations. Indeed, the smaller the size of the population, the larger the probability of the lost of diversity. Therefore, proper mechanisms for avoiding premature convergence must be applied.

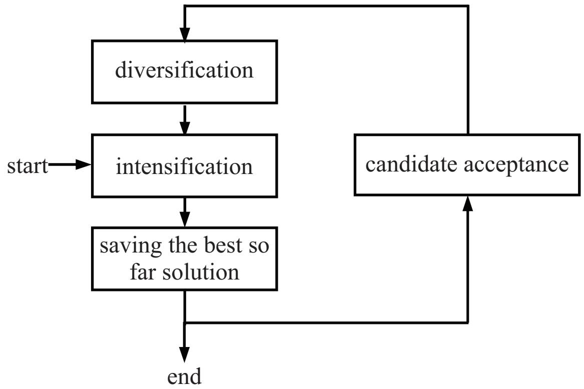


Figure 2. Basic flowchart of I&D

The principle that can help putting the above enhancements into practice is known as "intensification and diversification" (I&D) [7]. The idea behind is to seek high quality solutions by iterative applying local search ("recreation") and perturbation ("ruin") procedures. I&D is initiated by improvement of a starting solution. Then, one perturbs the existing solution (or its part). After that, the solution is again improved by means of a local search algorithm, and so on. I&D is distinguished for three main components: intensification, diversification, and candidate acceptance criterion (see Figure 2). The goal of intensification is to concentrate the search in a localized area (the neighbourhood of the current solution). Diversification is responsible for escaping regions of "attraction" in the search space. It may be achieved by certain perturbations of solutions (instead of saying "perturbation", other terms may be utilized: "mutation", "ruin", etc.). Finally, an acceptance criterion is used to decide which solution is actually chosen for the subsequent perturbation (see Section 2.4.3). I&D may be incorporated into the genetic-evolutionary algorithm in two ways.

▪ Firstly, I&D is applied to single solutions, in particular, to the offspring obtained by parent recombination. We call this post-recombination procedure the *intra-evolutionary process* (or simply *intra-evolution*). Intra-evolution could be seen as a very special case of the conventional evolutionary algorithm, that is, the population size is equal to 1, the reproduction scheme is "1, 1" (or "1 + 1"), and the single solution mutation procedure is used instead of the recombination operator. The mutation procedure plays, namely, the role of diversification. Regarding intensification, we found the tabu search (TS) technique [7] ideal for the GPP. The details of the TS procedure are described in Section 2.4.1.

▪ Secondly, I&D may also be applied for the populations. In this case, it is enough to think of the population as some kind of meta-solution; that is, instead of improvement/perturbation of the single solution, the meta-solution, i.e. population is to be improved/perturbed. Basically, this means that the whole population (or, at least, its part) undergoes some deep mutation. In particular, the test is performed at every generation, whether the diversity within the current population is below a given threshold. If it is, then a new population is created by applying mutations and following improvement; otherwise, the search is continued with the current population in an ordinary way. We call the above process the *inter-evolution*.

2. The improved genetic-evolutionary algorithm for the grey pattern problem

2.1. Creation of the initial population

The initial population (say, P) is created in two phases: firstly, $PS = |P|$ individuals, i.e. the GPP solutions are generated in a pure random way; secondly, all the solutions of the produced population are improved by the intra-evolutionary algorithm (see Section 2.4). As a result, the population is created that consists solely of locally optimal solutions. Eventually, the population members are sorted according to the increasing values of the objective function.

2.2. Selection mechanism

Two individuals are selected to be predecessors of a new recombined individual. For the selection, we apply a rank based rule [31]. In particular, the position of the first parent, u_1, within the sorted population is determined by the formula: $u_1 = \lfloor v^{\sigma} \rfloor$, where v is a uniform random number from the interval $[1, PS^{1/\sigma}]$, where PS is the population size, and σ is a real number in the interval $[1, 2]$ (it is referred to as a selection factor). The same formula is utilized for defining of the position for the second parent, u_2 ($u_2 \neq u_1$). It is obvious that the larger value of σ, the more probability that the better individual will be selected for recombination.

2.3. Recombination of solutions

Recombination of solutions remains one of the key factors by constructing competitive genetic/evolutionary algorithms. Very likely, the role of recombination operators within the hybrid algorithms is even more important. The classical recombination operators like one point [8] or uniform [28] operators are not very well fitted for the grey pattern problem. This is due to the fact that basically only the first m elements (black points) determine the solution of the GPP (see Figure 3). The interchange of any of the first m elements does not influence the objective function value (the same is true for the last $n - m$ elements (white points)).

Recombination is a structured process that ensures that the offspring will inherit the elements (genes) which are common to both parents. We can also think of recombination as a special sort diversification instrument; that is, it is desirable that recombination would add some randomness (certain elements, which are not contained in the parents, should be incorporated into the offspring). The degree of "disruptiveness" is introduced to formally describe a measure of how much recombination is randomized, i.e. how many "foreign genes" there are in the offspring (see also [17]). The degree of disruptiveness, ρ, is defined as follows:

$$\rho = \{i \mid i \le m \wedge \pi^{\circ}(i) \notin \{\pi'(1), \pi'(2), \ldots, \pi'(m)\} \wedge \pi^{\circ}(i) \notin \{\pi''(1), \pi''(2), \ldots, \pi''(m)\}\}, \tag{3}$$

where π', π'' are the parents and π° is the offspring.

There are two situations: 1) $\rho = 0$; 2) $\rho > 0$. We assume that the second variant is preferable to the first one in the environment of the robust hybrid algorithm. In our algorithm, disruptiveness is flexibly controlled by adding $\lfloor \xi m \rfloor$ foreign genes to the offspring, where ξ is a parameter (disruptiveness factor). The recombination procedure itself is quite specific. It is based on the merging process suggested in [3, 5]. The detailed pseudo-code of this recombination procedure (written in an algorithmic language) is presented in Figure 4. An illustrative example is shown in Figure 5.

It should be noted that the heuristic procedure based on tabu search is incorporated into the recombination process to partially improve the offspring. This procedure is performed on a restricted set of the elements of the offspring. The details of the improvement process are considered in the next section. Also, note that we apply the above recombination process more than once at one generation. In our implementation, the number of recombinations per generation is controlled by the parameter N_{recomb}.

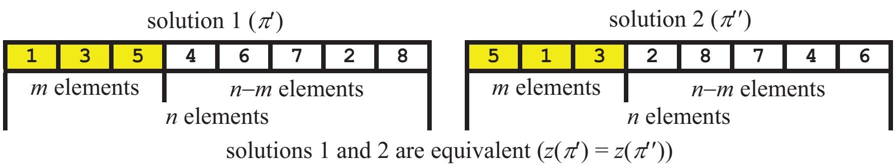
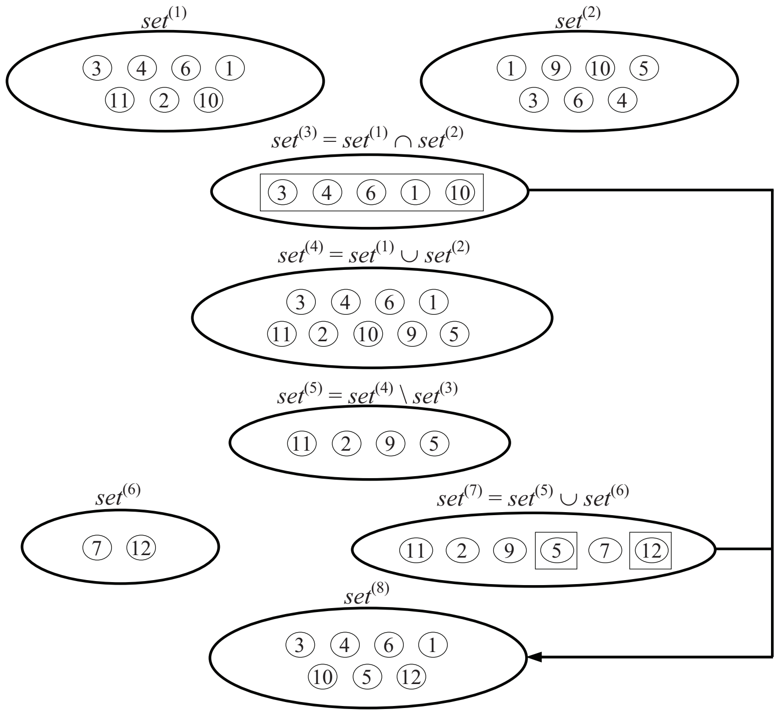


Figure 3. The case of solutions of the grey pattern problem

procedure Recombination;
// input: π', π'' – solutions-parents, n – problem size, m – number of black points, ξ_1, ξ_2 – disruptiveness factors
// output: π° – recombined solution (offspring)
$\rho_1 := \max\{1, \lfloor \xi_1 m \rfloor\}$; $\rho_2 := \max\{1, \lfloor \xi_2 m \rfloor\}$;
obtain $set^{(1)}$ from the first m elements of π'; // $|set^{(1)}| = m$
obtain $set^{(2)}$ from the first m elements of π''; // $|set^{(2)}| = m$
if not($set^{(1)} \equiv set^{(1)}$) **then begin** // the sets $set^{(1)}$ and $set^{(2)}$ are different
  $set^{(3)} := set^{(1)} \cap set^{(2)}$; $m^{intersect} = |set^{(3)}|$; $set^{(4)} := set^{(1)} \cup set^{(2)}$; // $m^{intersect} > 0$, $m^{union} = |set^{(4)}| = 2m - m^{intersect}$
  $set^{(5)} := set^{(4)} \setminus set^{(3)}$; // $m^{difference} = |set^{(5)}| = 2m - 2m^{intersect}$
  if $2m - m^{intersect} + \rho_1 \le n$ **then** $\rho := \rho_1$ **else** $\rho := n - 2m + m^{intersect}$;
  select ρ different elements from π' not in $set^{(4)}$ to form $set^{(6)}$;
  $set^{(7)} := set^{(5)} \cup set^{(6)}$; $m^{new} = |set^{(7)}|$; // $m^{new} = \min\{2m - 2m^{intersect} + \rho, n - m^{intersect}\}$
  add $m - m^{intersect}$ random elements from $set^{(7)}$ to $set^{(3)}$ to create $set^{(8)}$;
  // $set^{(8)}$ contains exactly m elements that serve as the starting offspring
  if $m^{new} - (m - m^{intersect}) > 0$ **then begin**
    obtain $\dot{\pi}$ from the elements of $set^{(8)}$; obtain $\ddot{\pi}$ from the elements of $set^{(7)}$ that are not in $set^{(8)}$;
    merge $\dot{\pi}$ and $\ddot{\pi}$ to get $\hat{\pi}$, the number of elements of $\hat{\pi}$ is equal to $m^{new} - m^{intersect}$;
    if $m^{new} - (m - m^{intersect}) \le 5$ **then** apply FastSteepestDescent to $\hat{\pi}$, get $\hat{\pi}$
      else apply FastRandomizedTabuSearch to $\hat{\pi}$, get $\hat{\pi}$
    // note: FastSteepestDescent and FastRandomizedTabuSearch are performed
    // only on the elements in $set^{(7)}$ and keeping the elements in $set^{(3)}$ fixed
  end
  else obtain $\hat{\pi}$ from the elements of $set^{(8)}$
end
else begin // the sets $set^{(1)}$ and $set^{(2)}$ are equivalent
  select ρ_2 different elements from π' not in $set^{(1)}$ to create $set^{(3)}$;
  remove ρ_2 random elements from $set^{(1)}$; obtain $\hat{\pi}$ from the elements of $set^{(1)}$ and $set^{(3)}$
end;
obtain $\bar{\pi}$ from the elements of set $\{1, 2, \ldots, n\} \setminus \{\hat{\pi}(1), \hat{\pi}(2), \ldots, \hat{\pi}(m)\}$;
merge $\hat{\pi}$ and $\bar{\pi}$ to get the offspring π°
end.

Figure 4. Pseudo-code of the recombination procedure for the GPP



$|set^{(1)}| = 7$, $|set^{(2)}| = 7$, $|set^{(3)}| = 5$, $|set^{(4)}| = 9$, $|set^{(5)}| = 4$, $|set^{(6)}| = 2$, $|set^{(7)}| = 6$, $|set^{(8)}| = 7$

Figure 5. Example of creating a starting offspring

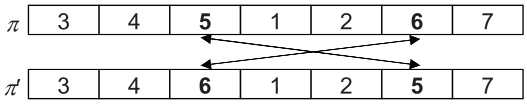


Figure 6. Example of the pairwise interchange

2.4. Local improvement (intra-evolution)

The post-recombination improvement is probably the most important component of IGEA. In the simplest case, we could utilize the ordinary descent local search for this purpose. But we should make use of more elaborated approaches like tabu search if we are seeking for superior quality results. The standard TS algorithms, however, suffer from cycling and stagnation phenomena. Fortunately, there exist the I&D approach (see Section 1). If intensification is performed, namely by means of the conventional TS, one gets the so-called iterated tabu search (ITS) method [20]. There are three main ingredients in the ITS approach: tabu search, mutation, and acceptance criterion.

2.4.1. Tabu search procedure

The central idea of tabu search is allowing climbing moves when no improving solution exists (this is in contrast to the descent local search, which terminates as soon as the locally optimal solution has been encountered). TS starts from an initial solution, and moves repeatedly from the current solution to a neighbouring one. We use the 2-exchange neighbourhood, N_2, where the neighbours are obtained by pairwise interchanges of the elements of a solution (see Figure 6). Formally, the neighbourhood N_2 of the solution π is defined by the formula: $N_2(\pi) = \{\pi^\nabla \mid \pi^\nabla = \pi \odot p_{ij}, \pi \in \Pi, i = 1,\dots,m, j = m+1,\dots,n\}$; here, $\pi^\nabla = \pi \odot p_{ij}$ means that π^∇ is obtained from π by applying the move p_{ij} (the move p_{ij} exchanges the ith and the jth element in the given permutation, i.e. $\pi^\nabla(i) = \pi(j) \wedge \pi^\nabla(j) = \pi(i)$).

At each step of TS, a set of the neighbours of the current solution is considered and the move that improves most the objective function value is chosen. If there are no improving moves, TS chooses one that least degrades the objective function. The reverse moves to the solutions just visited are to be forbidden in order to avoid cycling. The GPP allows implementing the list of tabu moves in an effective manner. In particular, the tabu list is organized as an integer matrix $\boldsymbol{T} = (t_{ij})_{m\times(n-m)}$. At the beginning, all the entries of $\boldsymbol{T}$ are set to zero. As the search progresses, the entry t_{ij} stores the current iteration number, plus the value of the tabu tenure, h, i.e. the number of the future iteration starting at which the ith and the jth elements may again be interchanged. In this case, an elementary perturbation (move) p_{ij} is tabu if the value of t_{ij} is equal or greater than the current iteration number. Note that testing whether a move is tabu or not requires only one comparison. We therefore call the above procedure the fast tabu search procedure.

An aspiration criterion allows permitting the tabu status to be ignored under favourable circumstances. Usually, the move from the solution π to solution $\pi^\Diamond$ is permitted (even if $\pi^\Diamond$ is tabu) if $z(\pi^\Diamond) < z(\pi^*)$, where π^* is the best solution found so far. The resulting decision rule looks thus as follows: replace the current solution π by the new solution $\pi^\blacklozenge$ such that $\pi^\blacklozenge = \arg\min_{\pi^\Diamond \in N_2^\Diamond(\pi)} z(\pi^\Diamond)$, where $N_2^\Diamond(\pi) = \{\pi^\Diamond \mid \pi^\Diamond \in N_2(\pi)$ and $((\pi^\Diamond$ is not tabu) or $(z(\pi^\Diamond) < z(\pi^*)))\}$.

TS forbids some moves from time to time. This fact means that certain portions of the search space are excluded from being visited. This can be seen as a disadvantage of the search process. One of the possible ways to get over this weakness is to minimize these restrictions, that is, it is desirable that the number of forbidden moves is as minimal as possible. We propose a very simple trick: the tabu status is disregarded with a small probability even if the aspiration criterion does not hold. We empirically found that the proper value of this probability, α, is somewhere between 0.05 and 0.1 (we used $\alpha = 0.05$). As the tabu status is ignored randomly with a negligible probability, there is almost no risk that the cycles will occur. This approach is called the randomized tabu search.

We also propose to include an additional component into the above TS procedure. Our idea is to embed an alternative intensification mechanism based on the deterministic steepest descent (SD) algorithm. The rationale of doing so is to prevent an accidental miss of a local optimum and to refine the search from time to time. Better results are achieved if the SD procedure is invoked at the moments of detecting improving solutions (that is, the inequality $z(\pi^\blacklozenge) - z(\pi) < 0$ holds). The alternative intensification procedure, however, is omitted if it already took place within the last ω steps (ω is an alternative intensification period (we used $\omega = \lfloor 0.03n \rfloor$)). The TS process continues until a termination criterion is satisfied (an a priori number of iterations, τ, have been performed).

We implemented two variants of the steepest descent procedure. The first one is simply based on searching in the 2-exchange neighbourhood N_2. The second one uses the extended 2-exchange neighbourhood N_2. The extended neighbourhood N_2 (denoted as $N_{2\oplus 2}$) can be described in the following way (also see [19]): $N_{2\oplus 2}(\pi) = N_2(\pi) \cup \{\pi^\wedge \mid \pi^\wedge \in \Pi, \pi^\wedge \neq \pi, \pi^\wedge = \pi^\blacksquare \odot p_{ij}, i = 1,\dots,m, j = m+1,\dots,n\}$, where $\pi^\blacksquare = \arg\min_{\pi^- \in N_2^-(\pi)} z(\pi^-)$, $N_2^-(\pi) = N_2(\pi) \setminus \{\arg\min_{\dot\pi \in N_2(\pi)} z(\dot\pi)\}$. The graphical interpretation of the neighbourhood $N_{2\oplus 2}$ is shown in Figure 7.

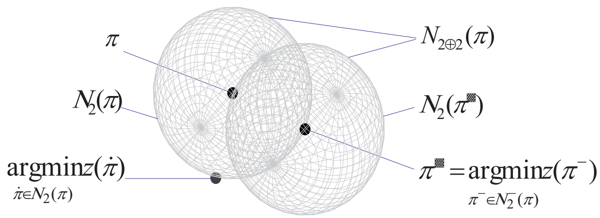


Figure 7. Graphical representation of the neighbourhood $N_{2\oplus 2}$

The pseudo-code of the randomized tabu search algorithm is presented in Appendix, Figure A1. The pseudo-codes of the steepest descent and extended steepest descent algorithms are given in Figures A2, A3.

Fast execution of the local improvement procedure is of high importance, as stated above. This is even more true for ITS where plenties of iterations of TS take place. Fortunately, lots of computations can be shortened due to very specific character of the matrix $\boldsymbol{A}$ of the GPP, as shown in [17, 29]. In particular, the exploration of the neighbourhood is restricted to the interchange of one of the first m elements (black points) with one of the last $n-m$ elements (white points). Consequently, the neighbourhood size decreases to $O(m(n-m))$ instead of $O(n^2)$ for the conventional QAP. Evaluating the difference in the objective function values thus becomes considerably faster. Instead of the standard formula, a simplified formula (4) is used (see also [17]):

$$\Delta z(\pi, i, j) = 2\sum_{k=1, k\neq i}^{m} (b_{\pi(j)\pi(k)} - b_{\pi(i)\pi(k)}),\quad i = 1, 2, ..., m,\ j = m+1, ..., n, \tag{4}$$

where $\Delta z(\pi, i, j)$ denotes the difference in the values of the objective function by interchanging the ith and the jth elements of the permutation π.

Drezner [5] proposed a very inventive technique which allows reducing the run time (CPU time) even more. Based on this technique, $\Delta z(\pi, i, j)$ is calculated according to the following formula [17]:

$$\Delta z(\pi, i, j) = 2(c_{\pi(j)} - c_{\pi(i)} - b_{\pi(i)\pi(j)}),\quad i = 1, 2, ..., m,\ j = m+1, ..., n, \tag{5}$$

where i and j denote the indices of the elements of the permutation and $c_{\pi(i)}$, $c_{\pi(j)}$ are the entries of an array C of size n. The entries of C are calculated once before starting the algorithm according to the formula: $c_i = \sum_{j=1}^{m} b_{i\pi(j)}, i = 1, 2, ..., n$ (here, π is the starting solution). So, this takes $O(mn)$ time. In case of moving from π to $\pi \odot p_{uv}$, updating of the values of c_i is performed according to the formula: $c_i = c_i + b_{i\pi(u)} - b_{i\pi(v)}$, which requires $O(n)$ time only. As the TS procedure is invoked many times, the overall effect is really surprising, especially if $m \ll n$.

2.4.2. Mutation

During the mutation process, the whole solution (or its part) is perturbed. At the first look, this is a relatively easy part of the ITS method. In fact, things are some more complicated. Mutations enable to escape local optima and allow discovering new and new regions of the search space. The mutation procedure for the GPP is based on random pairwise interchanges (RPIs) of certain elements of the given solution. The mutation process can be seen as a sequence of elementary perturbations $p_{r_1r_2}, p_{r_3r_4}, ...,$ $p_{r_{2\eta-1}r_{2\eta}}$; here, $p_{r_ir_{i+1}}$ denotes a random move which swaps the r_ith and the r_{i+1}th elements in the current permutation; thus, $\pi^{\sim} = \pi \odot p_{r_1r_2} \in N_2(\pi)$, $\pi^{\sim}(r_1) = \pi(r_2)$, $\pi^{\sim}(r_2) = \pi(r_1)$, $\pi^{\sim\sim} = \pi \odot p_{r_1r_2} \odot p_{r_3r_4} \in N_4(\pi)$ (if $r_1 \neq r_3$, $r_2 \neq r_4$), and so on.

All we need by implementing the RPI-mutation is to generate the couples of uniform random integers (r_i, r_{i+1}) such that $1 \leq r_i$, $r_{i+1} \leq n$, $i = 1, 3, ..., 2\eta$. The length of the sequence, η, is called the mutation level (strength). It is obvious that the larger the value of η, the stronger the mutation, and vice versa.

We can achieve more robustness if we let the parameter η vary in some interval, say $[\eta_{\min}, \eta_{\max}] \subseteq [1, n]$. The following strategy of changing the values of η may be proposed. At the beginning, η is equal to $\eta_{\min}$; further, η is increased gradually, step by step, until the maximum value $\eta_{\max}$ is reached; once $\eta_{\max}$ has been reached (or, possibly, a better local optimum has been found), the current value of η is immediately dropped to $\eta_{\min}$, and so on. In addition, if the best so far solution remains unchanged for a quite long time, then the value of $\eta_{\max}$ may be increased. The pseudo-code of the mutation procedure is presented in Appendix, Figure A4.

2.4.3. Acceptance criterion

The following are two main acceptance strategies: a) "exploitation", and b) "exploration". Exploitation is achieved by choosing only the currently best local optimum (the best so far solution). In case of exploration, each locally optimized solution (not necessary the best local optimum) can be considered as a potential candidate for perturbation. In IGEA, the so-called "where you are" (WYA) approach is applied — every new local optimum is accepted for diversification.

The pseudo-code of the resulting local improvement (intra-evolution) algorithm is presented in Figure 8.

```
procedure IntraEvolution;  // intra-evolution (post-recombination) process based on iterated tabu search
   // input: π – current solution, n – problem size, Q – # of iterations,  η_min, η_max – minimum and maximum mutation level
   // output: π* – the best solution found
   apply FastRandomizedTabuSearch to π, get improved solution π•;
   π := π•;  π* := π•;  η := η_min – 1;  // η is actual mutation level
   for q := 1 to Q do begin  // main cycle
      π := π•  // accept a solution for the subsequent mutation
      if η < η_max then η := η + 1 else η := η_min;  // update actual mutation level
      apply Mutation to selected solution π with mutation level η, obtain new solution π~;
      apply FastRandomizedTabuSearch to the solution π~, get new (improved) solution π•;
      if z(π•) < z(π*) then begin π* := π•; reset mutation level η, i.e. η := η_min – 1 end
   end  // for
end.
```

Figure 8. Pseudo-code of the intra-evolutionary algorithm

```
procedure ImprovedGeneticEvolutionaryAlgorithm;
   // input: PS – size of population, N_gen – # of generations, N_recomb – # of recombinations,
   //        Q – # of iterations of ITS, τ – # of iterations of TS, h_min, h_max – tabu tenures, σ – selection factor,
   //        ξ1, ξ2 – recombination disruptiveness factors, ζ1, ζ2 – intra-evolutionary mutation factors, ET – entropy threshold
   // output: π* – the best solution found
   StackHeader := 0; for i := 1 to m do for j := m + 1 to n do Tabu[i,j] := 0;  η_min := max{1,⌊ζ1 n⌋};  η_max := max{1,⌊ζ2 n⌋};
   create the locally optimized population P⊂Π in two steps: (i) generate initials solutions
     of P randomly, (ii) improve each member of P using intra-evolutionary algorithm;
     // note: increased number of the iterations of intra-evolution is used at this phase
   π* := argmin_{π∈P} {z(π)} ;  // π* denotes the best so far solution
   for generation := 1 to N_gen do begin  // main cycle
     sort the members of P in the ascending order of their quality;
     for recombined_solution := 1 to N_recomb do begin
       pick two solutions π′, π″ from P to be recombined;
       apply Recombination to π′ and π″, get recombined solution π°;
       apply IntraEvolution to π°, get improved solution π•;
       add π• to population P; if z(π•) < z(π*) then π* := π•
     end;  // for recombined_solution...
     cull population P by removing N_recomb worst individuals;
     if entropy of P is below ET then make hot restart in two steps: (i) mutate all the members
        of P, except the best one, (ii) improve each mutated solution using intra-evolution
   end  // for generation...
end.
```

Figure 9. Pseudo-code of the improved genetic-evolutionary algorithm

2. 5. Population replacement scheme

For the population replacement, we utilize the well known "$\mu + \lambda$" strategy. In this case, the individuals chosen at the end of the reproduction iteration are the best ones of $P_\mu \cup P_\lambda$, where P_μ is the population at the beginning of the reproduction, and P_λ denotes the set of newly created individuals (in our algorithm, $\mu = PS$, $\lambda = N_{\mathrm{recomb}}$). An additional replacement mechanism ("hot" restart) is activated if the loss of the diversity has been identified. The following are two main phases of the hot restart: a) mutation, b) local improvement (intra-evolution). As a hot restart criterion, we use a measure of entropy [6]. The normalized entropy of the population, E, is defined in the following way:

$$E = \sum_{i=1}^{n} e_i \Big/ \hat{E}\,, \tag{6}$$

where

$$e_i = \begin{cases} 0, \gamma_i = 0 \\ -\frac{\gamma_i}{PS}\log\frac{\gamma_i}{PS}, \text{otherwise} \end{cases}, \tag{7}$$

γ_i is the number of times that the ith element ($\pi(i)$) appears between the 1st and mth position in the current population. $\hat{E}$ denotes the maximal available entropy. It can be derived according to the following formula:

$$\hat{E} = \frac{\kappa(n-\nu)}{PS} L' + \frac{\kappa'\nu}{PS} L''\,, \tag{8}$$

where

$$L' = \begin{cases} 0, \kappa = 0 \\ -\log\frac{\kappa}{PS}, \text{otherwise} \end{cases}, \quad L'' = -\log\frac{\kappa'}{PS}\,, \tag{9}$$

where $\kappa = (m \times PS - 1)$ div n, $\kappa' = \kappa + 1$, $\nu = ((m \times PS - 1) \bmod n) + 1$ (here, x div $y = \lfloor x/y \rfloor$, $x \bmod y = x - \lfloor x/y \rfloor \times y$).

The normalized entropy E takes values between 0 and 1. So, if E is less than the predefined entropy threshold ET, we state that premature convergence (stagnation) takes place. In this case, the population undergoes the hot restart process. After the restart, the algorithm continues in a standard way.

The resulting pseudo-code of the improved hybrid genetic-evolutionary algorithm (IGEA) is given in Figure 9. Note that there are two versions of IGEA depending on the descent algorithm used in the intra-evolutionary (tabu search) algorithm (see Appendix, Figure A1): in IGEA^1, the pure steepest descent procedure is used, while IGEA^2 uses the extended steepest descent procedure.

3. Computational experiments

In this section, we present the results of the experimentation with the proposed genetic-evolutionary algorithm. In the experiments, we used the GPP instances generated according to the method described in [29]. For the set of problems tested, the size of the instances, n, is equal to 256, and the frames are of dimensions 16×16, i.e. $n_1 = n_2 = 16$. The instances are denoted by the name **16-16-**m, where m is the density of grey; it varies from 3 to 128. Remind that, for these instances, the data matrix $\boldsymbol{B}$ remains unchanged, while the matrix $\boldsymbol{A}$ is of the form $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where 1 is a sub-matrix of size $m \times m$ composed of 1s only [30].

We experimented with the following control parameter settings: $PS = 8$; $N_{gen} = 25$; $N_{recomb} = 1$; $Q = 3$; $\tau = \lfloor 0.1n \rfloor = 25$; $h_{min} = \lfloor 0.08n \rfloor$, $h_{max} = \lfloor 0.12n \rfloor$; $\sigma = 1.95$; $\xi_1 = 0.1$; $\xi_2 = 0.1$; $\zeta_1 = 0.1$, $\zeta_2 = 0.2$; $ET = 0.005$ (PS denotes population size, N_{gen} — # of generations, N_{recomb} — # of recombinations per generation, Q — # of iterations of the intra-evolutionary algorithm (the search depth), τ — # of iterations of the tabu search procedure (within the intra-evolutionary algorithm), h_{min}, h_{max} — lower and higher tabu tenures, σ — selection factor, ξ_1, ξ_2 — recombination disruptiveness factors, ζ_1, ζ_2 — intra-evolutionary mutation factors, ET — entropy threshold). These parameter values are identical to those used in [17]. In the experiments, 3 GHz Pentium computer was used.

Firstly, we have compared our algorithm (version IGEA^2) with an evolutionary algorithm (EA) of Drezner presented in [5]. We compared the average run time (CPU time) and the average deviation of the obtained solutions, where the average deviation, $\bar{\delta}$, is calculated by the formula: $\bar{\delta} = 100(\bar{z} - z^{\lozenge})/z^{\lozenge}$ [%]; here, $\bar{z}$ is the average objective function value over K runs ("cold restarts") of the algorithm and $z^{\lozenge}$ denotes the best known value (BKV) of the objective function. Different starting populations are used at each run. Both algorithms use fast objective function evaluation (as described in formulas (4), (5)). The number of runs (K) is equal to 100 for EA. We used $K = 10$. The results of the comparison are shown in Table 1.

Since our algorithm constantly finds the best known ((pseudo-)optimal) solutions, it is preferable to investigate run time performance instead of solution quality. In such situation, the so-called "time to target" [1] methodology may by used. In this case, for a given target value of the objective function (target solution), the run time of the algorithm to achieve this value is recorded. This is repeated multiple times and the recorded run times are sorted. With each run time, a probability $P_i = \dfrac{i - 0.5}{w}$ is associated, where i ($i = 1, 2, \ldots, w$) denotes the number of the current trial and w is the total number of trials (we used $w = 10$). The probabilities P_i can be visualized using "time-to-target" plots which show the probability that the target value will be obtained (see Figure 10).

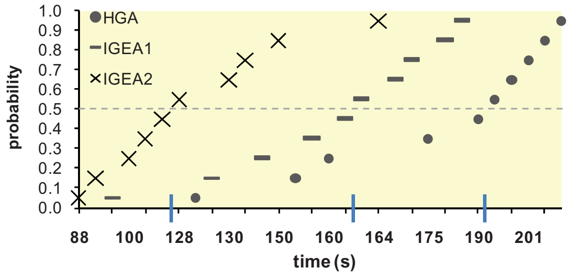


Figure 10. Example of the time-to-target plot for the instance **16-16-90**

The performance improvement factor, PIF, of one algorithm (A_1) to another one (A_2) may be defined by the following formula: $PIF = \dfrac{t_{0.5}(A_2)}{t_{0.5}(A_1)}$; here, $t_{0.5}$ denotes the time needed to obtain the target value with probability 0.5. We have compared the performance improvement factors for our improved genetic-evolutionary algorithms (IGEA^1, IGEA^2) and the previous hybrid genetic algorithm (HGA) presented in [17]. In Table 2, we present, in particular, the values of $t_{0.5}$ as well as the values of the performance improvement factor for HGA, IGEA^1 and IGEA^2. The values of $t_{0.5}$ are in seconds. The target values are set to be equal to the corresponding best known values (BKVs). (Only fifty-one instances (from $m = 65$ to $m = 115$) are examined because the remaining problems with $m < 65$ and $m > 115$ are, with few exceptions ($m = 26, 44, 45, 46$), easily solved by all algorithms.)

Our previous genetic algorithm (HGA) is very efficient and aggressive, so it is quite difficult to increase its overall performance. Nevertheless, with our new algorithms (IGEA^1 and IGEA^2), we have overcome its efficiency. This is especially true for the extended algorithm IGEA^2. In particular, for the examined instances, the performance improvement factors of IGEA^1 to HGA and IGEA^2 to HGA are equal to approximately 1.16 and 1.32, respectively (see Table 2).

Table 1. Results of the experiments with the GPP (I)

Instance	Best known value (BKV)	EA Av. dev.†	EA Time‡	IGEA² Av. dev.†	IGEA² Time‡
16-16-3	7810 [a]	n/a	n/a	0.000	0.0
16-16-4	15620 [a]	n/a	n/a	0.000	0.0
16-16-5	38072 [a]	n/a	n/a	0.000	0.0
16-16-6	63508 [a]	n/a	n/a	0.000	0.0
16-16-7	97178 [a]	n/a	n/a	0.000	0.0
16-16-8	131240 [a]	n/a	n/a	0.000	0.0
16-16-9	183744 [a]	n/a	n/a	0.000	0.0
16-16-10	242266 [a]	n/a	n/a	0.000	0.0
16-16-11	304722 [a]	n/a	n/a	0.000	0.1
16-16-12	368952 [a]	n/a	n/a	0.000	0.1
16-16-13	457504 [a]	n/a	n/a	0.000	0.1
16-16-14	547522 [a]	n/a	n/a	0.000	0.1
16-16-15	644036 [a]	n/a	n/a	0.000	0.1
16-16-16	742480 [a]	n/a	n/a	0.000	0.1
16-16-17	878888 [a]	n/a	n/a	0.000	0.2
16-16-18	1012990 [a]	n/a	n/a	0.000	0.1
16-16-19	1157992 [a]	n/a	n/a	0.000	0.2
16-16-20	1305744 [a]	n/a	n/a	0.000	0.3
16-16-21	1466210 [a]	n/a	n/a	0.000	0.5
16-16-22	1637794 [a]	0.000	67	0.000	0.3
16-16-23	1820052 [a]	0.000	67	0.000	0.2
16-16-24	2010846 [a]	0.000	56	0.000	0.6
16-16-25	2215714 [b]	0.001	97	0.000	3.2
16-16-26	2426298 [c]	0.021	100	0.000	16.5
16-16-27	2645436 [a]	0.006	100	0.000	1.1
16-16-28	2871704 [a]	0.040	107	0.000	0.9
16-16-29	3122510 [a]	0.001	94	0.000	0.7
16-16-30	3373854 [a]	0.000	92	0.000	0.5
16-16-31	3646344 [a]	0.055	84	0.000	0.6
16-16-32	3899744 [a]	0.124	76	0.000	0.5
16-16-33	4230950 [a]	0.004	59	0.000	0.7
16-16-34	4560162 [a]	0.019	61	0.000	2.6
16-16-35	4890132 [a]	0.006	104	0.000	3.2
16-16-36	5222296 [a]	0.005	105	0.000	2.0
16-16-37	5565236 [a]	0.000	101	0.000	1.8
16-16-38	5909202 [a]	0.000	91	0.000	0.9
16-16-39	6262248 [a]	0.000	67	0.000	1.1
16-16-40	6613472 [a]	0.001	108	0.000	0.9
16-16-41	7002794 [a]	0.001	137	0.000	0.6
16-16-42	7390586 [a]	0.001	143	0.000	0.7
16-16-43	7794422 [b]	0.001	199	0.000	3.2
16-16-44	8217264 [b]	0.107	201	0.000	16.0

Instance	Best known value (BKV)	EA Av. dev.†	EA Time‡	IGEA² Av. dev.†	IGEA² Time‡
16-16-45	8674910 [c]	0.018	206	0.000	150
16-16-46	9129192 [c]	0.005	162	0.000	64
16-16-47	9575736 [a]	0.004	78	0.000	3.1
16-16-48	10016256 [a]	0.021	106	0.000	2.0
16-16-49	10518838 [b]	0.018	181	0.000	3.4
16-16-50	11017342 [a]	0.000	193	0.000	2.8
16-16-51	11516840 [b]	0.000	207	0.000	7.5
16-16-52	12018388 [b]	0.002	194	0.000	6.3
16-16-53	12558226 [a]	0.001	193	0.000	4.6
16-16-54	13096646 [b]	0.004	175	0.000	4.0
16-16-55	13661614 [b]	0.010	295	0.000	10.1
16-16-56	14229492 [b]	0.005	234	0.000	2.8
16-16-57	14793682 [b]	0.000	167	0.000	2.2
16-16-58	15363628 [b]	0.005	216	0.000	2.3
16-16-59	15981086 [a]	0.005	235	0.000	3.5
16-16-60	16575644 [a]	0.039	238	0.000	2.4
16-16-61	17194812 [b]	0.021	386	0.000	2.2
16-16-62	17822806 [b]	0.003	100	0.000	3.6
16-16-63	18435790 [a]	0.003	381	0.000	1.9
16-16-64	19050432 [a]	0.028	516	0.000	2.3
16-16-65	19848790 [b]	0.019	416	0.000	3.1
16-16-66	20648754 [b]	0.013	237	0.000	4.5
16-16-67	21439396 [b]	0.028	305	0.000	9.7
16-16-68	22234020 [b]	0.059	244	0.000	18.0
16-16-69	23049732 [b]	0.025	284	0.000	27
16-16-70	23852796 [b]	0.079	256	0.000	26
16-16-71	24693608 [b]	0.034	286	0.000	78
16-16-72	25522408 [d]	n/a	n/a	0.000	490
16-16-73	26375828 [e]	0.057	335	0.000	298
16-16-74	27235240 [f]	0.062	358	0.000	304
16-16-75	28114952 [b]	0.020	343	0.000	41
16-16-76	29000908 [b]	0.010	319	0.000	121
16-16-77	29894452 [f]	0.016	376	0.000	145
16-16-78	30797954 [f]	0.013	302	0.000	117
16-16-79	31702182 [b]	0.022	253	0.000	11.6
16-16-80	32593088 [b]	0.058	178	0.000	3.3
16-16-81	33544628 [b]	0.004	94	0.000	3.9
16-16-82	34492592 [b]	0.002	124	0.000	70
16-16-83	35443938 [e]	0.000	167	0.000	57
16-16-84	36395172 [e]	0.001	178	0.000	61
16-16-85	37378800 [e]	0.001	206	0.000	151
16-16-86	38376438 [c]	0.000	162	0.000	94

Instance	Best known value (BKV)	EA Av. dev.†	EA Time‡	IGEA² Av. dev.†	IGEA² Time‡
16-16-87	39389054 [b]	0.000	127	0.000	25
16-16-88	40416536 [b]	0.014	132	0.000	23
16-16-89	41512742 [b]	0.006	173	0.000	183
16-16-90	42597626 [e]	0.006	302	0.000	165
16-16-91	43676474 [e]	0.007	321	0.000	224
16-16-92	44759294 [g]	0.010	224	0.000	157
16-16-93	45870244 [e]	0.005	260	0.000	214
16-16-94	46975856 [e]	0.016	273	0.000	190
16-16-95	48081112 [h]	0.026	217	0.000	169
16-16-96	49182368 [a]	0.065	236	0.000	216
16-16-97	50344050 [a]	0.040	155	0.000	213
16-16-98	51486642 [a]	0.052	145	0.000	188
16-16-99	52660116 [a]	0.020	169	0.000	201
16-16-100	53838088 [a]	0.005	109	0.000	117
16-16-101	55014262 [a]	0.012	125	0.000	84
16-16-102	56202826 [h]	0.012	96	0.000	40
16-16-103	57417112 [a]	0.002	82	0.000	73
16-16-104	58625240 [h]	0.008	115	0.000	62
16-16-105	59854744 [a]	0.001	103	0.000	38
16-16-106	61084902 [a]	0.002	113	0.000	33
16-16-107	62324634 [a]	0.001	122	0.000	21
16-16-108	63582416 [a]	0.000	179	0.000	12.6
16-16-109	64851966 [a]	0.000	87	0.000	11.1
16-16-110	66120434 [h]	0.000	235	0.000	10.7
16-16-111	67392724 [a]	0.000	358	0.000	8.2
16-16-112	68666416 [a]	0.001	790	0.000	7.7
16-16-113	69984758 [a]	n/a	n/a	0.000	10.2
16-16-114	71304194 [a]	n/a	n/a	0.000	6.3
16-16-115	72630764 [a]	n/a	n/a	0.000	5.1
16-16-116	73962220 [a]	n/a	n/a	0.000	5.3
16-16-117	75307424 [a]	n/a	n/a	0.000	4.0
16-16-118	76657014 [a]	n/a	n/a	0.000	3.6
16-16-119	78015914 [a]	n/a	n/a	0.000	2.3
16-16-120	79375832 [a]	n/a	n/a	0.000	1.7
16-16-121	80756852 [a]	n/a	n/a	0.000	1.6
16-16-122	82138768 [a]	n/a	n/a	0.000	1.4
16-16-123	83528554 [a]	n/a	n/a	0.000	1.0
16-16-124	84920540 [a]	n/a	n/a	0.000	0.7
16-16-125	86327812 [a]	n/a	n/a	0.000	0.4
16-16-126	87736646 [a]	n/a	n/a	0.000	0.3
16-16-127	89150166 [a]	n/a	n/a	0.000	0.2
16-16-128	90565248 [a]	n/a	n/a	0.000	0.2

† the average deviation (Av. dev.) is measured in percentage of average solution over BKV;
‡ times for EA and IGEA² are given in seconds per run for 2.8 GHz and 3 GHz computers, respectively;
[a] comes from [30]; [b] comes from [14]; [c] comes from [13]; [d] comes from this paper;
[e] comes from [5,17]; [f] comes from [15]; [g] comes from [27]; [h] comes from [16]

Table 2. Results of the experiments with the GPP (II)

Instance	$t_{0.5}^{◇}$	$t_{0.5}^{◈}$	$t_{0.5}^{◆}$	PIF_1	PIF_2	PIF_3
16-16-65	2.9	2.9	2.4	1.00	1.21	1.21
16-16-66	4.7	3.5	3.2	1.34	1.47	1.09
16-16-67	9.3	9.0	6.7	1.03	1.39	1.34
16-16-68	19.3	16.0	12.3	1.21	1.57	1.30
16-16-69	28.1	23.8	15.6	1.18	1.80	1.53
16-16-70	26.6	22.4	21.4	1.19	1.24	1.05
16-16-71	82.8	67.5	56.3	1.23	1.47	1.20
16-16-72	n/a	597.0	450.0	n/a	n/a	1.33
16-16-73	330.0	263.2	285.0	1.26	1.16	0.92
16-16-74	284.7	280.0	227.0	1.02	1.25	1.23
16-16-75	49.7	40.2	33.5	1.24	1.48	1.20
16-16-76	109.3	105.4	96.0	1.04	1.14	1.10
16-16-77	149.0	136.0	124.3	1.10	1.20	1.09
16-16-78	98.7	96.7	88.9	1.02	1.11	1.09
16-16-79	12.3	10.4	8.0	1.18	1.54	1.30
16-16-80	3.5	3.1	3.1	1.13	1.13	1.00
16-16-81	3.7	3.3	3.0	1.12	1.23	1.10

Instance	$t_{0.5}^{◇}$	$t_{0.5}^{◈}$	$t_{0.5}^{◆}$	PIF_1	PIF_2	PIF_3
16-16-82	61.1	59.0	51.4	1.04	1.19	1.15
16-16-83	55.4	47.4	48.0	1.17	1.15	0.99
16-16-84	61.0	52.2	49.5	1.17	1.23	1.05
16-16-85	132.6	131.8	110.3	1.01	1.20	1.19
16-16-86	93.8	93.8	70.1	1.00	1.34	1.34
16-16-87	20.9	18.1	16.1	1.15	1.30	1.12
16-16-88	25.2	23.5	20.5	1.07	1.23	1.15
16-16-89	172.0	163.0	140.1	1.06	1.23	1.16
16-16-90	191.3	160.7	127.0	1.19	1.51	1.27
16-16-91	246.0	189.8	171.0	1.30	1.43	1.10
16-16-92	172.8	156.0	132.6	1.11	1.30	1.18
16-16-93	267.7	195.1	203.5	1.37	1.32	0.96
16-16-94	196.9	150.0	170.9	1.31	1.15	0.88
16-16-95	216.0	145.1	150.0	1.49	1.44	0.97
16-16-96	239.8	215.1	157.0	1.11	1.53	1.37
16-16-97	216.7	211.0	180.2	1.03	1.20	1.17
16-16-98	190.0	180.4	134.5	1.05	1.41	1.34

Instance	$t_{0.5}^{◇}$	$t_{0.5}^{◈}$	$t_{0.5}^{◆}$	PIF_1	PIF_2	PIF_3
16-16-99	194.0	172.0	162.0	1.13	1.20	1.06
16-16-100	110.1	99.0	101.4	1.11	1.09	0.98
16-16-101	94.0	77.3	81.0	1.22	1.16	0.95
16-16-102	36.9	37.0	34.9	1.00	1.06	1.06
16-16-103	74.5	66.4	52.8	1.12	1.41	1.26
16-16-104	68.5	59.3	50.0	1.16	1.37	1.19
16-16-105	43.8	32.9	32.4	1.33	1.35	1.02
16-16-106	34.0	32.7	25.0	1.04	1.36	1.31
16-16-107	20.4	14.0	16.0	1.46	1.28	0.88
16-16-108	12.6	10.8	10.6	1.17	1.19	1.02
16-16-109	11.1	9.4	8.2	1.18	1.35	1.15
16-16-110	11.2	11.0	8.1	1.02	1.38	1.36
16-16-111	8.2	8.1	6.2	1.01	1.32	1.31
16-16-112	7.3	6.2	5.9	1.18	1.24	1.05
16-16-113	10.1	7.9	6.4	1.28	1.58	1.23
16-16-114	6.5	4.9	4.1	1.33	1.59	1.20
16-16-115	4.7	4.1	3.8	1.15	1.24	1.08
Average:				**1.16**	**1.32**	**1.15**

Notes: $t_{0.5}^{◇} = t_{0.5}(\mathrm{HGA})$, $t_{0.5}^{◈} = t_{0.5}(\mathrm{IGEA}^1)$, $t_{0.5}^{◆} = t_{0.5}(\mathrm{IGEA}^2)$, $PIF_1 = \frac{t_{0.5}(\mathrm{HGA})}{t_{0.5}(\mathrm{IGEA}^1)}$, $PIF_2 = \frac{t_{0.5}(\mathrm{HGA})}{t_{0.5}(\mathrm{IGEA}^2)}$, $PIF_3 = \frac{t_{0.5}(\mathrm{IGEA}^1)}{t_{0.5}(\mathrm{IGEA}^2)}$

Table 3. Results of the experiments with the GPP (III)

Instance	$t_{0.5}^{\circ}$	$t_{0.5}^{\bullet}$	$t_{0.5}^{\Diamond}$	PIF_4	PIF_5	PIF_6	Instance	$t_{0.5}^{\circ}$	$t_{0.5}^{\bullet}$	$t_{0.5}^{\Diamond}$	PIF_4	PIF_5	PIF_6	Instance	$t_{0.5}^{\circ}$	$t_{0.5}^{\bullet}$	$t_{0.5}^{\Diamond}$	PIF_4	PIF_5	PIF_6
16-16-65	*1.8*	*1.2*	*0.2*	1.50	9.00	6.00	**16-16-82**	*39.4*	*26.1*	*3.9*	1.51	10.10	6.69	**16-16-99**	*124.1*	*77.1*	*12.0*	1.61	10.34	6.43
16-16-66	*2.8*	*1.6*	*0.2*	1.75	14.00	8.00	**16-16-83**	*35.0*	*20.9*	*3.6*	1.67	9.72	5.81	**16-16-100**	*67.7*	*44.1*	*7.5*	1.54	9.03	5.88
16-16-67	*5.7*	*3.9*	*0.5*	1.46	11.40	7.80	**16-16-84**	*37.2*	*23.4*	*3.7*	1.59	10.05	6.32	**16-16-101**	*58.0*	*32.8*	*6.1*	1.77	9.51	5.38
16-16-68	*12.1*	*7.0*	*0.9*	1.73	13.44	7.78	**16-16-85**	*83.2*	*58.9*	*8.3*	1.41	10.02	7.10	**16-16-102**	*23.6*	*15.9*	*2.6*	1.48	9.08	6.12
16-16-69	*17.7*	*10.7*	*1.2*	1.65	14.75	8.92	**16-16-86**	*56.3*	*40.8*	*5.3*	1.38	10.62	7.70	**16-16-103**	*47.7*	*29.1*	*3.9*	1.64	12.23	7.46
16-16-70	*16.1*	*9.4*	*1.6*	1.71	10.06	5.88	**16-16-87**	*12.7*	*8.1*	*1.2*	1.57	10.58	6.75	**16-16-104**	*41.9*	*25.5*	*3.7*	1.64	11.32	6.89
16-16-71	*52.9*	*29.7*	*4.2*	1.78	12.60	7.07	**16-16-88**	*15.9*	*10.0*	*1.5*	1.59	10.60	6.67	**16-16-105**	*27.1*	*14.6*	*2.4*	1.86	11.29	6.08
16-16-72	*451.9*	*262.0*	*33.2*	1.72	13.61	7.89	**16-16-89**	*106.2*	*72.3*	*10.5*	1.47	10.11	6.89	**16-16-106**	*21.6*	*14.5*	*1.8*	1.49	12.00	8.06
16-16-73	*210.4*	*115.0*	*21.5*	1.83	9.79	5.35	**16-16-90**	*116.9*	*67.5*	*9.4*	1.73	12.44	7.18	**16-16-107**	*12.8*	*6.0*	*1.2*	2.13	10.67	5.00
16-16-74	*171.2*	*118.0*	*17.0*	1.45	10.07	6.94	**16-16-91**	*155.7*	*84.6*	*12.9*	1.84	12.07	6.56	**16-16-108**	*7.6*	*4.8*	*0.8*	1.58	9.50	6.00
16-16-75	*31.2*	*16.9*	*2.5*	1.85	12.48	6.76	**16-16-92**	*110.5*	*67.1*	*9.8*	1.65	11.28	6.85	**16-16-109**	*6.7*	*4.1*	*0.6*	1.63	11.17	6.83
16-16-76	*68.0*	*46.5*	*7.2*	1.46	9.44	6.46	**16-16-93**	*169.9*	*88.1*	*15.0*	1.93	11.33	5.87	**16-16-110**	*6.8*	*4.7*	*0.6*	1.45	11.33	7.83
16-16-77	*91.1*	*60.0*	*9.3*	1.52	9.80	6.45	**16-16-94**	*121.2*	*66.2*	*12.8*	1.83	9.47	5.17	**16-16-111**	*5.2*	*3.5*	*0.5*	1.49	10.40	7.00
16-16-78	*63.3*	*41.2*	*6.6*	1.54	9.59	6.24	**16-16-95**	*131.1*	*63.9*	*11.1*	2.05	11.81	5.76	**16-16-112**	*4.4*	*2.7*	*0.4*	1.63	11.00	6.75
16-16-79	*7.8*	*4.4*	*0.6*	1.77	13.00	7.33	**16-16-96**	*153.9*	*95.1*	*11.7*	1.62	13.15	8.13	**16-16-113**	*6.3*	*3.5*	*0.5*	1.80	12.60	7.00
16-16-80	*2.2*	*1.3*	*0.2*	1.69	11.00	6.50	**16-16-97**	*132.2*	*95.2*	*13.3*	1.39	9.94	7.16	**16-16-114**	*4.0*	*2.2*	*0.3*	1.82	13.33	7.33
16-16-81	*2.3*	*1.4*	*0.2*	1.64	11.50	7.00	**16-16-98**	*122.4*	*79.7*	*10.1*	1.54	12.12	7.89	**16-16-115**	*2.9*	*1.7*	*0.3*	1.71	9.67	5.67
Average:																		**1.65**	**11.09**	**6.76**

Notes: $t_{0.5}^{\circ} = t_{0.5}(\mathrm{SD})$, $t_{0.5}^{\bullet} = t_{0.5}(\mathrm{SA})$, $t_{0.5}^{\Diamond} = t_{0.5}(\mathrm{IGEA}^2)$, $PIF_4 = \dfrac{t_{0.5}(\mathrm{SD})}{t_{0.5}(\mathrm{SA})}$, $PIF_5 = \dfrac{t_{0.5}(\mathrm{SD})}{t_{0.5}(\mathrm{IGEA}^2)}$, $PIF_6 = \dfrac{t_{0.5}(\mathrm{SA})}{t_{0.5}(\mathrm{IGEA}^2)}$

Table 4. Results of the experiments with the GPP (IV)

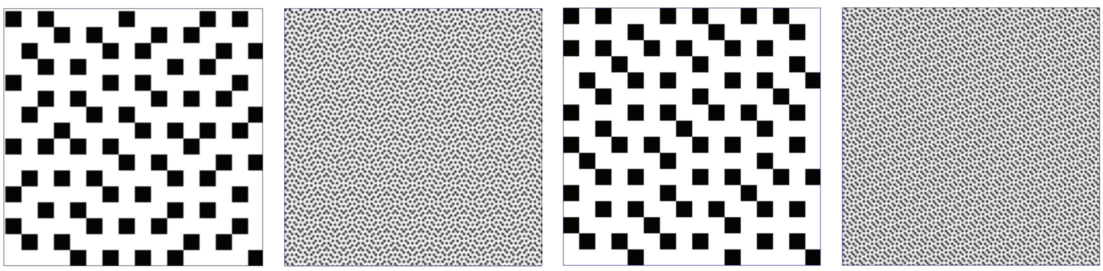
Instance	$t_{0.5}^{\square}$	$t_{0.5}^{\boxtimes}$	$t_{0.5}^{\Diamond}$	PIF_7	PIF_8	PIF_9	Instance	$t_{0.5}^{\square}$	$t_{0.5}^{\boxtimes}$	$t_{0.5}^{\Diamond}$	PIF_7	PIF_8	PIF_9	Instance	$t_{0.5}^{\square}$	$t_{0.5}^{\boxtimes}$	$t_{0.5}^{\Diamond}$	PIF_7	PIF_8	PIF_9
16-16-65	*0.6*	*0.4*	*0.2*	1.50	3.00	2.00	**16-16-82**	*12.3*	*7.7*	*3.9*	1.60	3.15	1.97	**16-16-99**	*41.0*	*21.0*	*12.0*	1.95	3.42	1.75
16-16-66	*1.0*	*0.4*	*0.2*	2.50	5.00	2.00	**16-16-83**	*11.2*	*5.9*	*3.6*	1.90	3.11	1.64	**16-16-100**	*23.2*	*13.0*	*7.5*	1.78	3.09	1.73
16-16-67	*1.9*	*1.1*	*0.5*	1.73	3.80	2.20	**16-16-84**	*12.4*	*6.6*	*3.7*	1.88	3.35	1.78	**16-16-101**	*18.7*	*10.1*	*6.1*	1.85	3.07	1.66
16-16-68	*3.8*	*2.1*	*0.9*	1.81	4.22	2.33	**16-16-85**	*27.3*	*17.3*	*8.3*	1.58	3.29	2.08	**16-16-102**	*7.8*	*4.6*	*2.6*	1.70	3.00	1.77
16-16-69	*5.7*	*3.1*	*1.2*	1.84	4.75	2.58	**16-16-86**	*19.8*	*11.5*	*5.3*	1.72	3.74	2.17	**16-16-103**	*15.0*	*8.5*	*3.9*	1.76	3.85	2.18
16-16-70	*5.5*	*2.9*	*1.6*	1.90	3.44	1.81	**16-16-87**	*4.3*	*2.3*	*1.2*	1.87	3.58	1.92	**16-16-104**	*13.7*	*7.5*	*3.7*	1.83	3.70	2.03
16-16-71	*16.8*	*9.0*	*4.2*	1.87	4.00	2.14	**16-16-88**	*5.3*	*2.9*	*1.5*	1.83	3.53	1.93	**16-16-105**	*8.9*	*4.0*	*2.4*	2.23	3.71	1.67
16-16-72	*141.1*	*74.1*	*33.2*	1.90	4.25	2.23	**16-16-89**	*34.5*	*21.4*	*10.5*	1.61	3.29	2.04	**16-16-106**	*7.0*	*4.2*	*1.8*	1.67	3.89	2.33
16-16-73	*69.4*	*33.9*	*21.5*	2.05	3.23	1.58	**16-16-90**	*38.9*	*21.1*	*9.4*	1.84	4.14	2.24	**16-16-107**	*4.0*	*1.8*	*1.2*	2.22	3.33	1.50
16-16-74	*57.4*	*36.4*	*17.0*	1.58	3.38	2.14	**16-16-91**	*51.9*	*25.2*	*12.9*	2.06	4.02	1.95	**16-16-108**	*2.6*	*1.4*	*0.8*	1.86	3.25	1.75
16-16-75	*9.9*	*5.1*	*2.5*	1.94	3.96	2.04	**16-16-92**	*35.6*	*20.4*	*9.8*	1.75	3.63	2.08	**16-16-109**	*2.3*	*1.2*	*0.6*	1.92	3.83	2.00
16-16-76	*22.1*	*13.4*	*7.2*	1.65	3.07	1.86	**16-16-93**	*53.5*	*24.9*	*15.0*	2.15	3.57	1.66	**16-16-110**	*2.4*	*1.4*	*0.6*	1.71	4.00	2.33
16-16-77	*29.8*	*17.9*	*9.3*	1.66	3.20	1.92	**16-16-94**	*39.0*	*19.0*	*12.8*	2.05	3.05	1.48	**16-16-111**	*1.7*	*1.1*	*0.5*	1.55	3.40	2.20
16-16-78	*20.0*	*12.0*	*6.6*	1.67	3.03	1.82	**16-16-95**	*42.9*	*17.8*	*11.1*	2.41	3.86	1.60	**16-16-112**	*1.5*	*0.8*	*0.4*	1.88	3.75	2.00
16-16-79	*2.6*	*1.3*	*0.6*	2.00	4.33	2.17	**16-16-96**	*49.2*	*27.8*	*11.7*	1.77	4.21	2.38	**16-16-113**	*2.1*	*1.0*	*0.5*	2.10	4.20	2.00
16-16-80	*0.7*	*0.4*	*0.2*	1.75	3.50	2.00	**16-16-97**	*43.0*	*27.9*	*13.3*	1.54	3.23	2.10	**16-16-114**	*1.4*	*0.6*	*0.3*	2.33	4.67	2.00
16-16-81	*0.8*	*0.4*	*0.2*	2.00	4.00	2.00	**16-16-98**	*40.2*	*22.9*	*10.1*	1.76	3.98	2.27	**16-16-115**	*1.0*	*0.5*	*0.3*	2.00	3.33	1.67
Average:																		**1.86**	**3.65**	**1.97**

Notes: $t_{0.5}^{\square} = t_{0.5}(\mathrm{TS})$, $t_{0.5}^{\boxtimes} = t_{0.5}(\mathrm{GA\text{-}SD})$, $t_{0.5}^{\Diamond} = t_{0.5}(\mathrm{IGEA}^2)$, $PIF_7 = \dfrac{t_{0.5}(\mathrm{TS})}{t_{0.5}(\mathrm{GA\text{-}SD})}$, $PIF_8 = \dfrac{t_{0.5}(\mathrm{TS})}{t_{0.5}(\mathrm{IGEA}^2)}$, $PIF_9 = \dfrac{t_{0.5}(\mathrm{GA\text{-}SD})}{t_{0.5}(\mathrm{IGEA}^2)}$

Table 5. New best known solution for the GPP

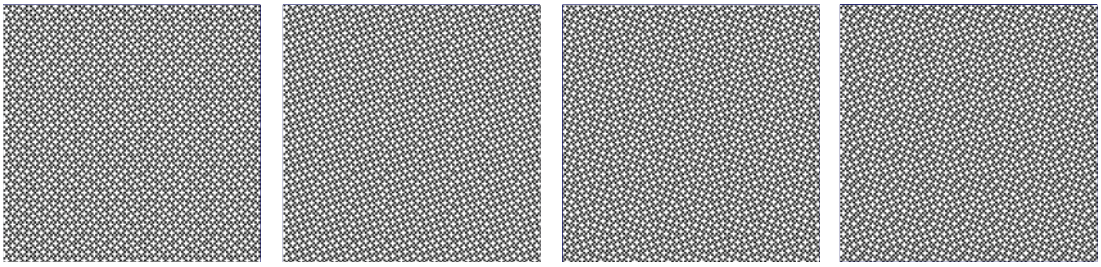
Instance	Previous best known value	New best known value
16-16-72	25529984[a]	**25522408**

[a] comes from [15]



(a) (b)

Figure 11. Previous (a) and new (b) best known grey frames of density 72/256: larger- and smaller-scale views



(a) (b) (c) (d)

Figure 12. (Pseudo-)optimal grey frames of densities 100/256 (a), 101/256 (b), 102/256 (c), 103/256 (d)

We have also compared the algorithm IGEA[2] with other well-known algorithms, in particular, steepest descent (SD) algorithm, simulated annealing (SA) algorithm, tabu search (TS) algorithm, and genetic (evolutionary) algorithm hybridized with steepest descent (GA-SD). All these algorithms were coded and implemented by the author; the descriptions of the algorithms can be found in [5]. The results of the comparison of the algorithms are presented in Tables 3 and 4. Similarly to Table 2, we present the values of $t_{0.5}$ and *PIF*. The values of $t_{0.5}$ are again in seconds, however the target values are 0.1% above BKVs.

Note that during the experiments, we were successful in discovering new record-breaking solution for the instance **16-16-72** ($m = \mathbf{72}$) (see Table 5). As a confirmation of the quality of the solution produced, we give the visual representation of this solution and the previous best known solution in Figure 11. Some other (pseudo-)optimal grey frames are shown in Figure 12 so that the reader can judge about the excellence of the grey patterns generated.

4. Conclusions

In this work, the issues related to solving the grey pattern problem (GPP) are discussed. We propose to use an improved genetic-evolutionary algorithm (IGEA), which is based on the integrating of intensification and diversification (I&D) approaches. The main improvements of IGEA are due the special recombination of solutions and the enhanced intra-evolutionary procedure as a post-recombination algorithm. The recombination has both diversification and intensification effect. The post-recombination algorithm itself consists of the iterative tabu search and mutation processes, where the tabu search serves as a basic intensification mechanism. The fast descent and extended decent-based local search procedures are designed to play the role of alternative intensification. The specialized mutation operator has exclusively diversification effect.

The new results from the experiments show promising performance of the proposed algorithm, as well as its superiority to the previous efficient hybrid genetic algorithm proposed in [17]. These results support the opinion that is extremely important to use a smart post-recombination procedure as well as a proper mechanism for premature convergence avoidance. It is confirmed that integrating I&D into the evolutionary process has a quite remarkable effect on the quality of solutions.

The effectiveness of our algorithm is also corroborated by the fact that all GPP instances are solved to pseudo-optimality at surprisingly small computational effort. The new best known grey pattern of density 72/256 has been discovered.

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Appendix

```
procedure FastRandomizedTabuSearch;
  // input: π – current solution, n – problem size, m – # of black points, B – distance matrix,
  //        h_min, h_max – lower and higher tabu tenures, τ – # of iterations, ω – alternative intensification period, α – randomization level
  // output: π• – the best solution found
  for i := 1 to StackHeader do Tabu[Stack_1[i], Stack_2[i]] := 0; // tabu list initialization
  for i := 1 to n do begin c_i := 0; for j := 1 to m do c_i := c_i + b_iπ(j) end; // initialization of C
  π• := π; k := 1; k′ := 1; improved := FALSE; choose h randomly between h_min and h_max;
  while (k ≤ τ) or (improved = TRUE) then begin // main cycle
    Δ_min := ∞;
    for i := 1 to m do // m(n – m) neighbours of π are considered
      for j := m + 1 to n do begin
        Δ := 2(c_π(j) – c_π(i) – b_π(i)π(j));
        forbidden := if((Tabu[i,j] ≥ k) and (random() ≥ α), TRUE, FALSE); aspired := if((z(π) + Δ < z(π•)) and forbidden), TRUE, FALSE);
        if ((Δ < Δ_min) and not(forbidden)) or aspired then begin Δ_min := Δ; u := i; v := j end
      end; // for
    if Δ_min < ∞ then begin
      π := π ⊙ p_uv ; for i := 1 to n do c_i := c_i + b_iπ(u) – b_iπ(v); // replace the current solution by the new one and update C
      Tabu[u,v] := k + h; // update tabu list (make the move p_uv tabu)
      StackHeader := StackHeader+ 1; Stack_1[StackHeader] := u; Stack_2[StackHeader] := v
    end; // if
    improved := if(Δ_min < 0, TRUE, FALSE);
    if improved and (k – k′ ≥ ω) then begin // switch to alternative intensification (depending on the version of IGEA)
      apply FastSteepestDescent | FastExtendedSteepestDescent to π; k′ := k
    end;
    if z(π) < z(π•) then π• := π; // save the best so far solution
    k := k + 1
  end // while
end.
```

Figure A1. Pseudo-code of the fast randomized tabu search algorithm. Notes. 1. The function if(x,y_1,y_2) returns y_1 if x = TRUE, otherwise it returns y_2. 2. The function random() returns a pseudo-random number uniformly distributed in [0, 1]

```
procedure FastSteepestDescent;
  // input: π – current solution, n – problem size, m – # of black points, B – distance matrix
  // output: π – resulting (improved) solution
  for i := 1 to n do begin c_i := 0; for j := 1 to m do c_i := c_i + b_iπ(j) end;
  repeat // cycle is repeated until local optimum is reached
    Δ_min := 0;
    for i := 1 to m do
      for j := m + 1 to n do begin Δ := 2(c_π(j) – c_π(i) – b_π(i)π(j)); if Δ < Δ_min then begin Δ_min := Δ; u := i; v := j end end;
    if Δ_min < 0 then begin // replace the current solution by the better one
      π := π ⊙ p_uv ; for i := 1 to n do c_i := c_i + b_iπ(u) – b_iπ(v)
    end // if
  until Δ_min ≥ 0
end.
```

Figure A2. Pseudo-code of the fast steepest descent algorithm using the neighbourhood N_2

```
procedure FastExtendedSteepestDescent;
  // input: π – current solution, n – problem size, m – # of black points, B – distance matrix
  // output: π – resulting (improved) solution
  for i := 1 to n do begin c_i := 0; for j := 1 to m do c_i := c_i + b_{iπ(j)} end;
  repeat // cycle is repeated until local optimum is reached
    Δ_min^(1) := ∞; Δ_min^(2) := ∞; u^(1) := 0; v^(1) := 0;
    for i := 1 to m do
      for j := m + 1 to n do begin
        Δ := 2(c_{π(j)} – c_{π(i)} – b_{π(i)π(j)});
        if Δ < Δ_min^(1) then begin Δ_min^(2) := Δ_min^(1); u^(2) := u^(1); v^(2) := v^(1); Δ_min^(1) := Δ; u^(1) := i; v^(1) := j end
        else if Δ < Δ_min^(2) then begin Δ_min^(2) := Δ; u^(2) := i; v^(2) := j end
      end; // for
    π' := π ⊙ p_{u^(1),v^(1)} ; π'' := π ⊙ p_{u^(2),v^(2)} ; c' := c; for i := 1 to n do c_i := c_i + b_{iπ''(u^(2))} – b_{iπ''(v^(2))} ;
    apply FastSteepestDescent to π'', get π''';
    if z(π') < z(π) or z(π''') < z(π) then begin
      if z(π') < z(π''') then for i := 1 to n do c_i := c'_i + b_{iπ'(u^(1))} – b_{iπ'(v^(1))} ;
      π := argmin(z(π'), z(π''')); // replace the current solution by a new (better) one
      better_solution_found := TRUE
    end
    else better_solution_found := FALSE
  until better_solution_found = FALSE
end.
```

Figure A3. Pseudo-code of the fast extended steepest descent algorithm using the neighbourhood $N_{2\oplus 2}$

```
procedure Mutation;
  // input: π – current solution, n – problem size, m – # of black points, B – distance matrix, η – mutation level (η > 1)
  // output: π – mutated solution
  for i := 1 to η do begin
    generate two numbers j and k randomly, uniformly, 1 ≤ j < k ≤ n;
    π := π ⊙ p_jk ; // interchange the jth and the kth elements in π
    // the objective function value, z, is recalculated in O(m) operations, i.e. z(π ⊙ p_jk) = z(π) + 2 Σ_{l=1, l≠j}^{m} (b_{π(k)π(l)} – b_{π(j)π(l)})
  end // for
end.
```

Figure A4. Pseudo-code of the mutation procedure