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Reconfigurable Control of Two-Time Scale Systems in Presence of Additive Faults

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Abstract. This work presents an adaptive approach for fault tolerant control of singularly perturbed systems, where both actuator and sensor faults are examined in presence of external disturbances. For sensor faults, an adaptive controller is designed based on an output-feedback control scheme. The feedback controller gain is determined in order to stabilize the closed-loop system in the fault free case and vanishing disturbance, while the additive gain is updated using an adaptive law to compensate for the sensor faults and the external disturbances. To correct the actuator faults, a state-feedback control method based on adaptive mechanism is considered. The both proposed controllers depend on the singular perturbation parameter ε leading to ill-conditioned problems. A well-posed problem is obtained by simplifying the Lyapunov equations and subsequently the controllers using the singular perturbation method and the reduced subsystems yielding to an ε-independent controller. The control scheme, designed based on the Lyapunov stability theory, guarantees asymptotic stability in presence of additive faults and external disturbances provided the singular perturbation parameter is sufficiently small. Finally, a numerical example is presented to demonstrate the effectiveness of the obtained results.

Keywords: Reconfigurable control; Singularly perturbed systems; Time scale decomposition; Adaptive control; Sensor fault; Actuator fault; Lyapunov equations.

1. Introduction

Systems, where slow and fast dynamic phenomena arise, are called singularly perturbed systems. They model many control systems like robotic systems, motor control systems, chemical processes, convection-diffusion systems and electric circuits. Those systems are distinguished by the existence of a small positive parameter called singular perturbation parameter. It indicates the degree of separation between "slow" and "fast" modes of the system. Such small parameters can be used for modeling machine reactance in power systems, capacitance in electronic and wire inductance control systems. Singular perturbation parameter leads frequently to ill-conditioned results in the system analysis and synthesis methods. In order to handle such problems, a reduction technique, called singular perturbation method, is proposed in the literature. Thus, the full-order system is decoupled into slow and fast subsystems, which makes possible to deal with lower-order systems and consequently to simplify the analysis and synthesis problems [1-3]. The complexity of the singularly perturbed systems makes them vulnerable to faults being able to corrupt the controller, the sensors, the process itself or the actuators. In this case, an adequate control scheme, known as reconfigurable control or fault-tolerant control, is needed to provide system stability even in presence of defects. Otherwise, the occurrence of such faults may cause production to stop and threats human and material safety. The main purpose of the fault-tolerant control is to maintain stability and performance in the event of malfunctions in sensors, actuators or other system components and to inhibit the restrictions of conventional feedback control [4-7]. Fault-tolerant design methods can be classified into active and passive approaches. The active control technique requires a fault diagnosis block to detect and identify the faults in real time, and then a mechanism to adapt the controller to the new faulty situation according to the information recovered about malfunctions [8-11]. In contrast, the passive approach, also called reliable control, offers a fixed-parameter controller in order to maintain (at least) stability in occurrence of presumed faults without the need for controller reconfiguration and fault diagnosis scheme [6,9,11,12]. Diverse methods are proposed to design reconfigurable controllers in order to ensure closed-loop stability, even in faulty case. A reliable fault-tolerant control method designed for linear systems using simultaneous stabilization is carried out by Stoustrup and Blondel in [12]. Tellili *et al.* in [13] developed a reliable H_∞ controller for linear time-invariant multi-parameter singularly perturbed system to tolerate sensor faults and to ensure H_∞ performance. The full-order system controller is then simplified to three reduced reliable H_∞ sub-controllers based on the fast and slow problems through the manipulation of the algebraic Riccati equations. Liu *et al.* in [14] proposed a H_∞ fault tolerant controllers to overcome actuator faults. The authors handle three cases of such faults: normal, loss of efficiency, and outage. The designed H_∞ controllers can reduce the degree of conservatism compared with existing methods. Richter and Lunze in [15] developed the fault-hiding approach for linear and Hammerstein systems and used virtual actuators and sensors respectively, for actuator and sensor faults. The results are extended to piecewise affine systems in [8]. Another widely studied method consists in the design of fault tolerant control scheme based on adaptive control principles. Such approaches can be applied without using control restructuring and fault diagnosis procedures [16,17]. Many authors were interested in this subject. In particular, Chen and Saif in [17] designed an adaptive scheme to diagnose and to accommodate actuator faults in linear multi-input single-output (MISO) systems with unknown system parameters. The fault tolerant control problem is resolved using the remaining operative actuators. In [18], a class of adaptive control methods based on output feedback approach in order to stabilize linear systems with complete actuator failures is developed. Jin and Yang in [19] designed a direct adaptive state feedback control approach based on Lyapunov stability theory. The resulting closed-loop system is then asymptotically stable in the presence of actuator faults and external disturbances. The method is extended and improved in [20]. Wang et al. in [21] developed an adaptive output feedback control to accommodate actuator faults including outage, loss of efficiency and stuck. An adaptive fault tolerant controller including a fault estimation error minimization problem is designed by Casavola and Garone in [22]. The considered fault in this case is assumed to be piecewise constant with a slowly varying behavior. For the singularly perturbed systems, their control in presence of actuator failures is investigated by some authors. Liu in [14] developed a controller scheme for singularly perturbed systems in presence of actuator saturation under the assumption that the fast subsystem is stable. Xin et al. in [23,24] proposed the reduced-order adjoint systems, by which some methods to estimate the basin of attraction of singularly perturbed systems were developed. In [25], an optimal controller based on state-feedback approach is designed to control non-linear singularly perturbed systems subject to actuator saturation. However, the above-mentioned approaches were limited to actuator saturation and did not consider the loss of efficiency by sensors and actuators.

The main goal of this paper is to design an adaptive fault tolerant control scheme for singularly perturbed systems in presence of external disturbances and additive faults characterized by a loss of effectiveness in sensors and actuators. The remaining part of this work is organized as follows. The system description and preliminaries are presented in Section 2. The fault model and the control problem are formulated in case of sensor faults in Section 3. In Section 4, an adaptive fault tolerant control approach against actuator faults and external disturbances is established. An example of application is developed in Section 5 followed by a conclusion in the last section.

2. System description and preliminaries

Consider the following time-invariant two-time scales singularly perturbed system under external disturbances described by

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \varepsilon\,\dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_{1x} \\ B_{1z} \end{bmatrix} w(t) \\ \qquad + \begin{bmatrix} B_{2x} \\ B_{2z} \end{bmatrix} u(t) \\ y(t) = C \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \end{cases} \tag{1}$$

where $x \in \Re^{n1}$ and $z \in \Re^{n2}$ are state vectors, $u \in \Re^{m}$ is the control vector, $y \in \Re^{l}$ is the output, $w \in \Re^{q}$ models piecewise continuous bounded external disturbances acting on the system and verifying $\|w\| \leq \overline{w}$ with $\overline{w}$ being an unknown positive constant. $A_{11} \in \Re^{n1xn1}$, $A_{12} \in \Re^{n1xn2}$, $A_{21} \in \Re^{n2xn1}$, $A_{22} \in \Re^{n2xn2}$, $B_{1x} \in \Re^{n1xq}$, $B_{2x} \in \Re^{n1xm}$, $B_{1z} \in \Re^{n2xq}$, $C_1 = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} \in \Re^{lxn1}$, $C_2 = \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} \in \Re^{lxn2}$ and $B_{2z} \in \Re^{n2xm}$ are constant matrices. The matrix A_{22} is assumed to be nonsingular (standard singularly perturbed systems). The parameter ε, called singular perturbation parameter, is a positive scalar taking values between 0 and 1. Denote throughout the paper: $B_i = \begin{bmatrix} B_{ix}^T & B_{iz}^T \end{bmatrix}^T$ for $i = 1, 2$ and $\|(.)\|$ the Euclidian norm of (.).

According to the time-scale property of the singularly perturbed system, the slow and the fast subsystems of full-order system (1) can be derived by formally setting the singular perturbation parameter ε to zero [2,26]. The slow subsystem is obtained as

$$\begin{cases} \dot{x}_s = A_s x_s + B_{1s} w_s + B_{2s} u_s \\ y_s = C_s x_s + D_s u_s \end{cases} \quad (2)$$

where $A_s = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $C_s = C_1 - C_2 A_{22}^{-1} A_{21}$, $D_s = -C_2 A_{22}^{-1} B_2$, $B_{is} = B_{ix} - A_{12}A_{22}^{-1}B_{iz}$ for $i = 1, 2$; x_s, u_s, and w_s are respectively, the slow part of the states, the control input u, and the disturbance input w.

If (A_s, B_{2s}) is stabilizable and (C_s, A_s) is detectable, then there exists a symmetric and positive definite matrix P_s satisfying the following slow Lyapunov equation

$$(A_s + B_{2s}K_sC_s)^T P_s + P_s (A_s + B_{2s}K_sC_s) = -Q_s \quad (3)$$

where Q_s is any given positive definite symmetric matrix and K_s is a static output feedback gain stabilizing the slow subsystem such that $u_s = K_s\, y_s$. The closed-loop slow subsystem is then defined by

$$\dot{x}_s = (A_s + B_{2s}K_sC_s)\, x_s + B_{1s} w_s \ . \quad (4)$$

Most often in literature [13,27,28], the following approximation is used: $A_s = A_{11}$, $B_{1s} = B_{1x}$, $B_{2s} = B_{2x}$ and $C_s = C_1$. Consequently, the slow subsystem (2), the slow Lyapunov equation (3) and the closed-loop slow subsystem (4) can be approached respectively, by the equations (5), (6) and (7):

$$\dot{x}_s = A_{11} x_s + B_{1x} w_s + B_{2x} u_s \quad (5)$$

$$(A_{11} + B_{2x}K_sC_1)^T P_s + P_s (A_{11} + B_{2x}K_sC_1) = -Q_s \quad (6)$$

$$\dot{x}_s = (A_{11} + B_{2x}K_sC_1)\, x_s + B_{1x} w_s \ . \quad (7)$$

Since C is of full rank, it is assumed, without loss of generality, that the output matrix can be transformed to block-diagonal form [29], $C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$, where $C_1 x$ and $C_2 z$ describe respectively, the slow and the fast part of the output.

The fast subsystem is given by

$$\begin{cases} \varepsilon\, \dot{z}_f = A_{22} z_f + B_{1z} w_f + B_{2z} u_f \\ y_f = C_2 x_f \end{cases} \quad (8)$$

where z_f, u_f and w_f are the fast parts respectively, of the states, the control input u and the disturbance input w. The fast subsystem can be rewritten in the stretching (fast) time scale t/ε as follows

$$\begin{cases} \dot{z}_f = A_{22} z_f + B_{1z} w_f + B_{2z} u_f \\ y_f = C_2 x_f \end{cases} \quad (9)$$

Assuming that (A_{22}, B_{2z}) is stabilizable and (C_2, A_{22}) is detectable, there exists a symmetric and positive definite matrix P_f satisfying the following fast Lyapunov equation:

$$(A_{22} + B_{2z}K_fC_2)^T P_f + P_f (A_{22} + B_{2z}K_fC_2) = -Q_f \quad (10)$$

where Q_f is any given positive definite symmetric matrix and K_f is a static output feedback gain stabilizing the fast subsystem such that $u_f = K_f\, y_f$. The closed-loop fast subsystem is then defined by:

$$\dot{z}_f = (A_{22} + B_{2z}K_fC_2)\, z_f + B_{1z} w_f \ . \quad (11)$$

3. Controller design with respect to sensor fault

This section will concentrate on the design of an adaptive fault tolerant controller to handle sensor faults in presence of external disturbances.

3.1. Failure model and problem formulation

The sensor faults which have been taken into account in this section involve loss of efficiency. For the output y_i, $i = 1, .., l$, let y_i^f be the signal issued from the i-th faulty sensor; accordingly, the sensor-fault model is expressed as follows,

$$y_i^f(t) = \rho_i\, y_i(t), \quad (12)$$

where ρ_i represents the sensor efficiency factor and verifies $0 \le \underline{\rho_i} \le \rho_i \le \overline{\rho}_i \le 1$. $\underline{\rho_i}$ and $\overline{\rho}_i$ indicate the known lower and upper bounds of ρ_i, respectively. The case $\underline{\rho_i} = \overline{\rho}_i = 0$ describes the completely interruption of the sensor i. $\underline{\rho_i} > 0$ denotes the case of partial failure of y_i. If $\underline{\rho_i} = \overline{\rho}_i = 1$, then $y_i^f(t) = y_i(t)$, which depicts the case of no failure. Denoting $\rho = diag(\rho_i)$, $i = 1, .., l$, the uniform sensor-fault model becomes:

$$y^f(t) = \rho\, y(t). \quad (13)$$

The assumption that the control signals and disturbances use identical channels is used by many authors [19, 21] to solve robust control problems. Consequently, the following supposition will be held: $B_{1x} = B_{2x}\, F$ and $B_{1z} = B_{2z}\, F$, where F is a matrix with appropriate dimension.

The problem under consideration is to design a control law such that the closed-loop singularly perturbed system is asymptotically stable for any $\varepsilon \in\ ]0, \varepsilon^*]$ despite the sensor fault occurrence and disturbance.

3.2. Controller proposal for the full-order singularly perturbed system

Let us introduce the following notations

$$X = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \; A(\varepsilon) = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{bmatrix},$$

$$B_i(\varepsilon) = \begin{bmatrix} B_{ix} \\ \frac{B_{iz}}{\varepsilon} \end{bmatrix} \text{ for } i = 1, 2.$$

System (1) can be expressed as

$$\begin{cases} \dot{X}(t) = A(\varepsilon)\,X(t) + B_1(\varepsilon)\,w(t) \\ \qquad + B_2(\varepsilon)\,u(t) \\ y(t) = C\,X(t) \end{cases} \tag{14}$$

By using the sensor-fault model (13) and the hypothesis assumed for the disturbances, system (14) is transformed to

$$\begin{cases} \dot{X}(t) = A(\varepsilon)\,X(t) + B_2(\varepsilon)F\,w(t) \\ \qquad + B_2(\varepsilon)\,u(t) \\ y(t) = \rho C\,X(t) \end{cases} \tag{15}$$

The proposed fault tolerant controller to stabilize the system (14) is described by:

$$u(t) = K_1\,y(t) + K_2(t) \tag{16}$$

where K_1 is chosen such that $(A(\varepsilon) + B_2(\varepsilon)K_1C)$ is Hurwitz and $K_2(t)$ is governed by the following update law

$$K_2(t) = -\frac{B_2^T(\varepsilon)P(\varepsilon)X}{\left\|B_2^T(\varepsilon)P(\varepsilon)X\right\|}\,(\hat{k}_3(t) + \left\|K_1CX\right\|) \tag{17}$$

where $\hat{k}_3$ is adjusted using the following adaptive law

$$\frac{d\hat{k}_3(t)}{dt} = \varepsilon\,\gamma\left\|X^TP(\varepsilon)B_2(\varepsilon)\right\| \tag{18}$$

where γ is a suitable positive constant and ε is the singular perturbation parameter. Let $\tilde{k}_3(t) = \hat{k}_3(t) - k_3$. Then the expression (18) can be transformed to $\dot{\tilde{k}}_3(t) = \varepsilon\,\gamma\left\|X^TP(\varepsilon)B_2(\varepsilon)\right\|$.

As mentioned in [20], to avoid discontinuity which can be caused by the term $\left\|X^TP(\varepsilon)B_2(\varepsilon)\right\|$ in the case where $X = 0$, it is sufficient to add a small constant in the denominator of the control law (17).
The following theorem is proposed to solve the fault tolerant control problem (15):

Theorem 1: *Consider the singularly perturbed system described by equation (14). Suppose the following assumptions are satisfied,*

1. There exists a singular perturbation parameter $\varepsilon^* > 0$ such that $(A(\varepsilon), B_2(\varepsilon))$ is stabilizable and $(C, A(\varepsilon))$ is detectable for all $\varepsilon \in]0, \varepsilon^*]$;
2. There exists a symmetric and positive definite matrix $P(\varepsilon)$ satisfying the following Lyapunov equation :

$$(A(\varepsilon) + B_2(\varepsilon)\,K_1C)^T\,P(\varepsilon) + P(\varepsilon)\,(A(\varepsilon) + B_2(\varepsilon)\;K_1C) = -Q \tag{19}$$

where Q is any given positive definite symmetric matrix;

3. K_1 is designed such that $(A(\varepsilon) + B_2(\varepsilon)\;K_1C)$ is Hurwitz.

Then the fault tolerant controller (16) verifying the adaptive laws (17) and (18), stabilizes asymptotically the system (14) subject to sensor fault (13) and external disturbances for any $\varepsilon \in]0, \varepsilon^*]$.

Proof. From equations (15) and (16), it follows for the closed-loop fault tolerant control system:

$$\begin{cases} \dot{X}(t) = A(\varepsilon)\,X(t) + B_2(\varepsilon)F\,w(t) \\ \; + B_2(\varepsilon)\,K_1\rho C\,X(t) + B_2(\varepsilon)\,K_2(t) \\ y(t) = \rho C\,X(t) \end{cases} \tag{20}$$

which can be rewritten as:

$$\begin{aligned} &\dot{X}(t) = (A(\varepsilon) + B_2(\varepsilon)\,K_1C)\,X(t) + \\ &B_2(\varepsilon)\,K_1(\rho - I)\;CX(t) + B_2(\varepsilon)\,K_2(t) \\ &+ B_2(\varepsilon)\,F\,w(t) \end{aligned} \tag{21}$$

Define an ε-dependent Lyapunov function candidate,

$$V(\varepsilon) = X^T\,P(\varepsilon)\,X + \varepsilon^{-1}\,\gamma^{-1}\,\tilde{k}_3^2 > 0. \tag{22}$$

Computing the derivative of $V(\varepsilon)$ along the trajectories of system (21) and taking into account the assumptions about the disturbances leads to

$$\begin{aligned} &\dot{V}(\varepsilon) = X^T[(A(\varepsilon) + B_2(\varepsilon)\;K_1C)^T\,P(\varepsilon) \\ &\; + P(\varepsilon)\,(A(\varepsilon) + B_2(\varepsilon)\;K_1C)]\,X \\ &+ 2\,\varepsilon^{-1}\gamma^{-1}\tilde{k}_3\,\dot{\tilde{k}}_3 + 2\,X^TP(\varepsilon)B_2(\varepsilon)\,K_2 \\ &+ 2\,X^TP(\varepsilon)B_2(\varepsilon)\,F\,w \\ &+ 2X^TP(\varepsilon)B_2(\varepsilon)K_1\,(\rho - I)CX \end{aligned} \tag{23}$$

Let k_3 be a constant used to limit the unknown bounded constant $\overline{w}$ such that [21]

$$\left\|X^TP(\varepsilon)B_2(\varepsilon)\right\|\left\|F\right\|\overline{w} \le \left\|X^TP(\varepsilon)B_2(\varepsilon)\right\|\,k_3. \tag{24}$$

Since $\overline{\rho}_i \le 1$, it can be shown that

$$X^TP(\varepsilon)B_2(\varepsilon)K_1\,(\rho - I)CX \le \left\|K_1CX\right\|\,\left\|X^TP(\varepsilon)B_2(\varepsilon)\right\|. \tag{25}$$

Substituting the adaptive law (17) and the expression (19) into the equation (23) yields the following form

$$\dot{V}(\varepsilon) \le -2\,X^T Q\,X + 2\left\|X^T P(\varepsilon) B_2(\varepsilon)\right\| k_3 + 2\left\|K_1 C X\right\| \left\|X^T P(\varepsilon) B_2(\varepsilon)\right\| - 2\,\frac{X^T P(\varepsilon) B_2(\varepsilon) B_2^T(\varepsilon) P(\varepsilon) X}{\left\|B_2^T(\varepsilon) P(\varepsilon) X\right\|}\,(\hat{k}_3(t) + \left\|K_1 C X\right\|) + 2\,\varepsilon^{-1}\gamma^{-1}\tilde{k}_3\,\dot{\tilde{k}}_3 \tag{26}$$

This can be transformed into the following form,

$$\dot{V}(\varepsilon) \le -2\,X^T Q X + 2\left\|X^T P(\varepsilon) B_2(\varepsilon)\right\| (k_3 - \hat{k}_3(t) + \left\|K_1 C X\right\| - \left\|K_1 C X\right\|) + 2\,\varepsilon^{-1}\gamma^{-1}\tilde{k}_3\,\dot{\tilde{k}}_3 \tag{27}$$

In the light of the adaptation law (18), it is easy to see that $\dot{V}(\varepsilon) \le 0$. Therefore, the faulty closed-loop system is asymptotically stable for any singular perturbation parameter $\varepsilon \in]0, \varepsilon^*]$.

3.3. Controller simplification

The fault tolerant controller (16) involves an output feedback gain K_1 designed such that $(A(\varepsilon) + B_2(\varepsilon) K_1 C)$ is Hurwitz. Under mild technical conditions, asymptotic stability of the full-order singularly perturbed system is guaranteed through the asymptotic stability of both the slow subsystem and fast subsystem for sufficiently small values of the singular perturbation parameter ε [2,30].

The conception of the static output feedback gain K_1 can be achieved through the simultaneous design of static output feedback controllers for the reduced slow and fast subsystems for small ε and some constraints on the system [3,30,31]. Thus, a simplified ε-independent feedback controller is designed to stabilize the full-order system.

The approximations taken in Section 2 (see equations (5) and (9)) permit to simplify $B_2(\varepsilon)$ in the control law (17) through $B_2 = \begin{bmatrix} B_{2x} \\ B_{2z} \end{bmatrix}$. The ε-dependent $P(\varepsilon)$ will be simplified in the following section.

3.4. Solving Lyapunov equation using reduced order models

In order to alleviate the numerical stiffness caused by the simultaneous occurrence of slow and fast phenomena and characterized by the presence of the small singular perturbation parameter ε, the full-order Lyapunov equation (19) will be formulated using slow and fast subsystem components.

The structure of $P(\varepsilon)$ is assumed to be of the form

$$P(\varepsilon) = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}. \tag{28}$$

The solution $P(\varepsilon)$ is ε-dependent, because equation (19) contains ε^{-1}-order matrices. Let Q be of the form $Q = \begin{bmatrix} I_{n1} & 0 \\ 0 & \varepsilon^{-1} I_{n2} \end{bmatrix}$ where I is identity matrix and $K_1 = [K_{11} \;\; K_{12}]$. Expanding the Lyapunov equation (19) after the substitution of $P(\varepsilon)$ leads to the following partitioned three equations

$$a_{11}^T P_1 + \varepsilon^{-1} a_{21}^T P_2^T + P_1\,a_{11} + \varepsilon^{-1} P_2\,a_{21} = -I_{n1} \tag{29}$$

$$\varepsilon^{-1} a_{21}^T P_3 + a_{11}^T P_2 + P_1\,a_{12} + \varepsilon^{-1} P_2\,a_{22} = 0 \tag{30}$$

$$a_{12}^T P_2 + \varepsilon^{-1} a_{22}^T P_3 + P_2^T a_{12} + \varepsilon^{-1} P_3\,a_{22} = -\varepsilon^{-1} I_{n2} \tag{31}$$

where $a_{11} = A_{11} + B_{2x} K_{11} C_1$, $a_{12} = A_{12} + B_{2x} K_{12} C_2$, $a_{21} = A_{21} + B_{2z} K_{11} C_1$ and $a_{22} = A_{22} + B_{2z} K_{12} C_2$.

Pre- and post-multiplying equations (29) and (31) by the singular perturbation parameter ε and letting ε → 0 leads to the following zero-order equations

$$a_{11}^T P_1 + P_1\,a_{11} = -I_{n1} \tag{32}$$

$$a_{22}^T P_3 + P_3\,a_{22} = -I_{n2} \tag{33}$$

$$a_{21}^T P_2^T + P_2\,a_{21} = 0. \tag{34}$$

Since K_{11} and K_{12} can be approximated, respectively, by K_s and K_f [29,31], it is clear that equations (32) and (33) correspond, respectively, to the Lyapunov equations of the slow subsystem approximation described by equations (5) and the fast subsystem given by equation (9) of the singularly perturbed system (1). Hence, P_1 and P_3 correspond, respectively, to the solution P_s of the slow Lyapunov equation (3) (approximated by equation (6)) and the solution P_f of the fast Lyapunov equation (10). It follows that the solution $P(\varepsilon)$ of the equation (19) can be approximated by $P(\varepsilon) \approx \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}$. Thus, the ill-defined controller (16) is simplified and made ε-independent using the corresponding reduced order well defined problem.

4. Controller design with respect to actuator fault

In this section, a fault tolerant controller will be designed to compensate actuator fault and external disturbances and then the singular perturbation method will be used to simplify the ε-dependent controller to avoid the numerical stiffness caused by the presence of time scales.

4.1. Preliminaries and failure model

Consider the LTI singularly perturbed system (1) with its slow subsystem (2) and fast subsystem (9).

If (A_s, B_{2s}) is stabilizable, then the state feedback controller $u_s = G_s x_s$ is considered to stabilize the slow subsystem (2). G_s is the slow controller gain satisfying the following slow Lyapunov equation

$$(A_s + B_{2s} G_s)^T L_s + L_s (A_s + B_{2s} G_s) = -M_s \tag{35}$$

with M_s any given positive definite symmetric matrix. The closed-loop slow subsystem is then obtained

$$\dot{x}_s = (A_s + B_{2s}G_s)\,x_s + B_{1s}w_s \;. \quad (36)$$

Employing the same approximation used in Section 2 for the system matrices of the slow subsystem, the slow Lyapunov equation (35) and the closed-loop slow subsystem (36) can be approximated, respectively, by the equations (37) and (38):

$$(A_{11} + B_{2x}G_s)^T L_s + L_s\,(A_{11} + B_{2x}G_s) = -M_s \quad (37)$$

$$\dot{x}_s = (A_{11} + B_{2x}G_s)\,x_s + B_{1x}w_s \quad (38)$$

Assuming that (A_{22}, B_{2z}) is stabilizable, there exists a feedback controller $u_f = G_f z_f$ stabilizing the fast subsystem (9). G_f is the fast controller gain satisfying the following fast Lyapunov equation,

$$(A_{22} + B_{2z}G_f)^T L_f + L_f\,(A_{22} + B_{2z}G_f) = -M_f \quad (39)$$

with M_f any given positive definite symmetric matrix. The closed-loop fast subsystem is then described by

$$\dot{z}_f = (A_{22} + B_{2z}G_f)\,z_f + B_{1z}w_f \;. \quad (40)$$

The faults dealt with in this section are given by a decrease in effectiveness. For the control input u_i, $i = 1,..,m$, consider u_i^F the signal from the faulty actuator; consequently, the failure model is adopted as follows

$$u_i^F = \alpha_{ai}\,u_i \quad (41)$$

where α_{ai} represents the actuator efficiency factor and verifies $0 \le \underline{\alpha}_{ai} \le \alpha_{ai} \le \overline{\alpha}_{ai} \le 1$. $\underline{\alpha}_{ai}$ and $\overline{\alpha}_{ai}$ indicate the known lower and upper bounds on α_{ai}, respectively. Thus, if $\underline{\alpha}_{ai} = \overline{\alpha}_{ai} = 1$, the ith actuator is working perfectly, whereas if $\underline{\alpha}_{ai} > 0$, partial loss of effectiveness is present. The case $\underline{\alpha}_{ai} = \overline{\alpha}_{ai} = 0$ describes the completely failing of the actuator i. Denoting $\alpha_a = diag(\alpha_{ai})$, $i = 1,..,m$, the uniform actuator-fault model becomes,

$$u^F(t) = \alpha_a\,u(t) \quad (42)$$

The problem under consideration is to develop a fault tolerant controller making the closed-loop singularly perturbed system asymptotically stable in presence of actuator faults and external disturbances.

4.2. Controller synthesis for the global singularly perturbed system

Considering the system (14), the actuator fault (42) and the assumption held for the disturbances, system (14) can be rewritten as

$$\dot{X}(t) = A(\varepsilon)\,X(t) + B_2(\varepsilon)\,F\,w(t) + B_2(\varepsilon)\,\alpha_a\,u(t) \quad (43)$$

The proposed controller model to stabilize the system (14) under actuator faults and external disturbances is given by:

$$u(t) = G_1(t)\,X(t) + G_2(t) \quad (44)$$

where G_1 is chosen such that $(A(\varepsilon) + B_2(\varepsilon)G_1)$ is Hurwitz and $G_2(t)$ is designed using the following update law:

$$G_2(t) = -\frac{\beta\,B_2^T(\varepsilon)L(\varepsilon)X}{\alpha\left\|B_2^T(\varepsilon)L(\varepsilon)X\right\|}\,(\hat{k}(t) + \|G_1X\|) \quad (45)$$

where α and β are appropriate positive constants satisfying [19-21]

$$\alpha\left\|B_2^T(\varepsilon)L(\varepsilon)X\right\|^2 \le \beta\left\|B_2^T(\varepsilon)L(\varepsilon)X\sqrt{\alpha_a}\right\|^2 \quad (46)$$

and $\hat{k}$ obeys the following adaptive law:

$$\frac{d\hat{k}(t)}{dt} = \varepsilon\,\gamma\left\|X^T L(\varepsilon)B_2(\varepsilon)\right\| \quad (47)$$

where γ is an appropriate positive constant and ε is the singular perturbation parameter. Let $\tilde{k}(t) = \hat{k}(t) - k$, where k is a constant verifying

$$\left\|X^T L(\varepsilon)B_2(\varepsilon)\right\|\left\|F\right\|\overline{w} \le \left\|X^T L(\varepsilon)B_2(\varepsilon)\right\|\,k \quad (48)$$

then the update law (47) can be expressed as

$$\dot{\tilde{k}}(t) = \varepsilon\,\gamma\left\|X^T L(\varepsilon)B_2(\varepsilon)\right\| \quad (49)$$

As main result, the following theorem will be proposed to solve the fault tolerant control problem (43),

Theorem 2: *Consider the system described by equation (14). Suppose that:*

1. There exists a singular perturbation parameter $\varepsilon^* > 0$ such that $(A(\varepsilon), B_2(\varepsilon))$ is stabilizable for all $\varepsilon \in]0, \varepsilon^*]$;
2. For any given positive definite symmetric matrix M, there exists a symmetric and positive definite matrix $L(\varepsilon)$ satisfying the following Lyapunov equation :

$$(A(\varepsilon) + B_2(\varepsilon)G_1)^T\,L(\varepsilon) + L(\varepsilon)(A(\varepsilon) + B_2(\varepsilon)\;G_1) = -M \quad (50)$$

3. G_1 is chosen such that $(A(\varepsilon) + B_2(\varepsilon)\;G_1)$ is Hurwitz.

Then there exists $\varepsilon^* > 0$ such that for every $\varepsilon \in]0, \varepsilon^*]$, the controller described by equation (44), with the update laws (45) and (47) and the controller gain G_1, stabilize asymptotically the system (43) subject to actuator fault (42) and external disturbances.

Proof. The Lyapunov-based proof of stability can be shown using the following Lyapunov function candidate

$$V(\varepsilon) = X^T\,L(\varepsilon)\,X + \varepsilon^{-1}\gamma^{-1}\tilde{k}^2 > 0. \quad (51)$$

The other steps are similar to theorem 1 and are detailed in [32].

4.3. Design of the ε-independent controller

G_1 represents the state-feedback controller gain stabilizing the full-order singularly perturbed system (1) in absence of faults and perturbations. It is well known [2,33] that G_1 can be designed using the locale feedback gains G_s and G_f provided that (A_s, B_s) and (A_f, B_f) are controllable and A_{22}^{-1} exists. Accordingly, the composite control is of the form $u(t) = G_1(t)\,X(t) = [G_{11} \quad G_{12}]\begin{bmatrix} x \\ z \end{bmatrix}$ where $G_{12} = G_f$ and $G_{11} = (I_f + G_f\,A_{22}^{-1}\,B_2)G_s + G_f\,A_{22}^{-1}\,A_{21}$. Thus, the simplification of the feedback controller gain G_1 conserves the stability of the full-order singularly perturbed system. To simplify the adaptive laws (45) and (47), it is adequate to use the slow parts of the command matrix $B_2(\varepsilon)$ and the simplified form of the Lyapunov matrix $L(\varepsilon)$ which will be derived in the following.

4.4. Simplifying of the Lyapunov equation

To remove the numerical stiffness in the Lyapunov equation given by expression (50), the latter will be decomposed in slow and fast parts [34]. The structure of L(ε) is assumed to be of the form $L(\varepsilon) = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}$. The solution $L(\varepsilon)$ is dependent on the singular perturbation parameter ε because equation (50) contains ε^{-1}-order matrices. A positive definite symmetric matrix M will be chosen of the form $M = \begin{bmatrix} I_{n1} & 0 \\ 0 & \varepsilon^{-1} I_{n2} \end{bmatrix}$, where I is identity matrix. Substituting $L(\varepsilon)$ into Lyapunov equation (50) yields the following partitioned three equations

$$a_{11}^T L_1 + \varepsilon^{-1} a_{21}^T L_2^T + L_1\,a_{11} + \varepsilon^{-1} L_2\,a_{21} = -I_{n1} \qquad (52)$$

$$\varepsilon^{-1} a_{21}^T L_3 + a_{11}^T L_2 + L_1\,a_{12} + \varepsilon^{-1} L_2\,a_{22} = 0 \qquad (53)$$

$$a_{12}^T L_2 + \varepsilon^{-1} a_{22}^T L_3 + L_2^T a_{12} + \varepsilon^{-1} L_3\,a_{22} = -\varepsilon^{-1} I_{n2} \qquad (54)$$

where $a_{11} = A_{11} + B_{2x}\,G_{11}$, $a_{12} = A_{12} + B_{2x}\,G_{12}$, $a_{21} = A_{21} + B_{2z}\,G_{11}$ and $a_{22} = A_{22} + B_{2z}\,G_{12}$.

Using the same method as in Section 2 yields the following equations

$$a_{11}^T L_1 + L_1\,a_{11} = -I_{n1} \qquad (55)$$

$$a_{22}^T L_3 + L_3\,a_{22} = -I_{n2} \qquad (56)$$

$$a_{21}^T L_2^T + L_2\,a_{21} = 0 \qquad (57)$$

It is well known that the controller gains G_{11} and G_{12} can be approximated, respectively, by the slow controller gain G_s and the fast controller gain G_f [35]. Hence, expressions (55) and (56) match respectively, the slow and the fast Lyapunov equations until $M_s = I_{n1}$ and $M_f = I_{n2}$. Furthermore, considering equation (57), the approximate solution of the Lyapunov matrix becomes $L(\varepsilon) \approx \begin{bmatrix} L_s & 0 \\ 0 & L_f \end{bmatrix}$, which removes the numerical stiffness in the solution of equation (50).

5. Example of application

To illustrate the effectiveness of the proposed method, the following numerical example is given below.

5.1. Fault tolerant control in presence of sensor faults and disturbance

Consider the singularly perturbed system (1) with parameters given by

$$A_{11} = \begin{bmatrix} -5 & 0.2 \\ -0.5 & 6 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0.1 \\ -1 & 1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -9 & -8 \\ 0.3 & 0.1 \end{bmatrix},\ A_{22} = \begin{bmatrix} -7 & 1 \\ -0.5 & -6 \end{bmatrix},$$

$$B_2 = B_1 = \begin{bmatrix} 1 & 1.55 \\ 1 & -0.5 \\ 0.9 & 0.8 \\ 0.2 & 0.11 \end{bmatrix},\ C = \begin{bmatrix} -2 & 0.5 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}$$

and $w = \begin{bmatrix} 0.01\sin(5t) \\ 0.5 \end{bmatrix}$.

The full-order system is open loop unstable (the system has one positive pole). The considered faulty model is a 50% loss of effectiveness in the first sensor and 70% in the second sensor, that is, $\alpha_a = diag(0.5, 0.7)$. The singular perturbation parameter is set to $\varepsilon = 0.05$. In the fault free case, the output feedback controller is computed using the method described in [30] based on the reduced order models. The states trajectories starting from $X_0 = [0.2, 0.03, 0.02, 0.01]^T$, the output and the simplified controller gain are shown in Fig. 1. It is clear from the above mentioned figure that the full-order system

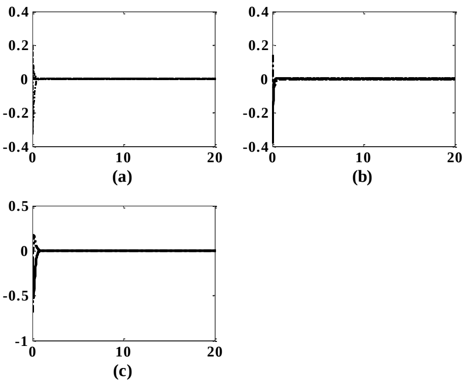

Figure 1. (a) States, (b) Output and (c) Controller $u(t)$ in the fault-free case by output feedback control and by ε = 0.05.

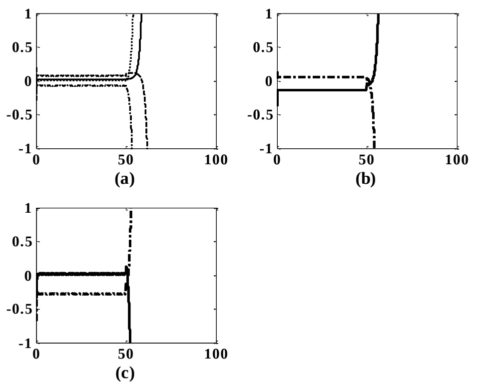

Figure 2. (a) States, (b) Output and (c) Controller $u(t)$ in the faulty case by output feedback control and by ε = 0.05.

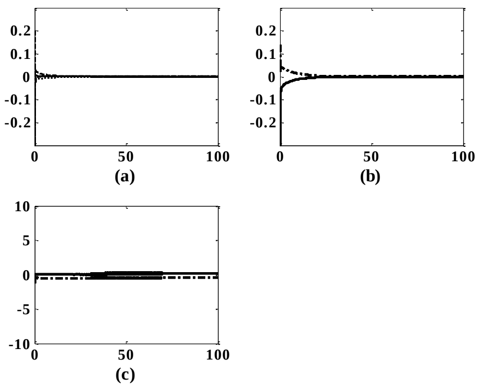

Figure 3. (a) States, (b) Output and (c) Controller $u(t)$ in the faulty case by adaptive fault tolerant control and by ε = 0.05.

is stabilized using the output feedback controller. However, the occurrence of sensor faults at time instance 50 sec causes instability of the full-order system (see Fig. 2). Next, an adaptive fault tolerant controller will be designed in presence of sensor faults and external disturbances. The simulation results are shown in Fig. 3. They indicate that the designed adaptive fault tolerant controller stabilizes asymptotically the singularly perturbed system subject to sensor faults and external disturbances.

Next, the case of actuator faults will be examined.

5.2. Fault tolerant control in presence of actuator faults and disturbance

The same singularly perturbed system treated below is considered with 10% loss of efficiency in the first actuator and 50% in the second actuator, that is, $\rho = \text{diag}(0.1, 0.5)$. In the fault free case, the full-order system is stabilized through a composite controller based on the slow and fast subsystems (see Fig. 4). The appearance of defects at time 50 sec yields a loss of the actuators performances which is indicated in Fig. 5. To compensate the fault effect, an adaptive fault tolerant controller is designed. Fig. 6 shows the trajectories of the states, the output, the controller $u(t)$ and the gain $k_3(t)$ after fault occurrence at time 50 sec by initial values $X_0 = [0.2, 0.03, 0.02, 0.01]^T$ and $\varepsilon = 0.05$. It can be deduced that the computed adaptive controller compensates the actuator faults in presence of external disturbances.

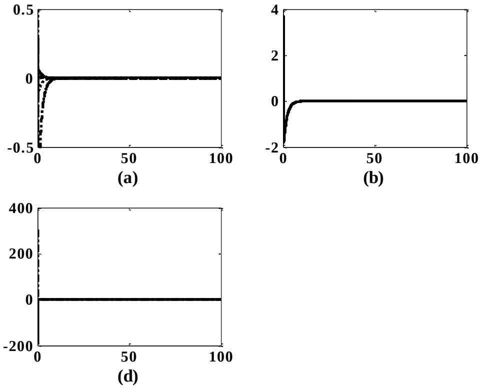

Figure 4. (a) States, (b) Output and (c) Controller $u(t)$ in the fault-free case by composite control

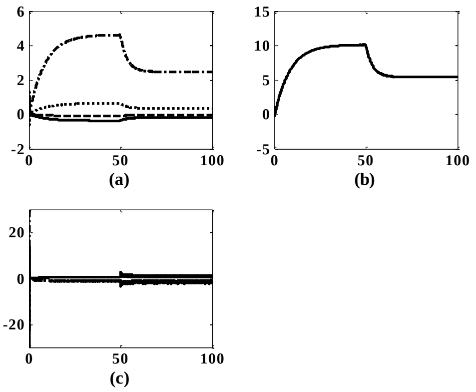

Figure 5. (a) States, (b) Output and (c) Controller $u(t)$ in the Faulty case by composite control

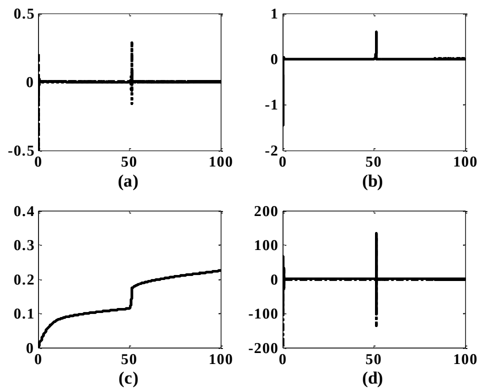

Figure 6. (a) States, (b) Output, (c) Gain $k_3(t)$ and (d) Controller $u(t)$ in the faulty case by adaptive ftc control

6. Conclusion

In this paper, the stabilization problem of singularly perturbed systems subject to additive faults and external disturbances is investigated. ε-dependent controllers are first proposed to handle sensor and actuator faults covering the normal operation and faulty cases. In both cases, the control scheme includes a feedback controller to handle the fault free case and an adaptive part to compensate additive faults and external disturbances. The resulting Ill-conditioned Lyapunov equation in the two cases is solved using the singular perturbation method and the simultaneous design of Lyapunov equations of the slow and fast subsystems. In the case of sensor faults, the first part of the controller, designed by an output feedback controller, is approximated by gains stabilizing the reduced order systems. While, in occurrence of actuator faults, the state-feedback controller is approximated by a composite controller which depends on the gains stabilizing the slow and fast subsystems. In both cases, the adaptive laws are simplified using ε-independent matrices. The synthesis of the reconfigurable control system guarantees the desired asymptotic stability of the full-order singularly perturbed system not only in the fault-free case, but also in occurrence of additive faults and external disturbances. The determination of the upper bound of singular perturbation parameter ε remains, as perspective, an important thematic.

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