On Speedy Recognition Of Non-Aliased Realization After Multifold Downsampling of an Oversampled Bandlimited Signal

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Abstract. The aim of the given paper is the development of the criterion and some expressions for recognizing a non-aliased realization in the set of realizations obtained by multifold decimation (f ltering and downsampling) of any oversampled bandlimited signal that has been obtained at the beginning by periodic sampling of a continuous-time signal. For each non-decimated as well as decimated realization discrete-time Fourier series coeff cient values, located at Nyquist frequency are calculated, using speedy recursive expressions based on reverse order processing of the given realizations. In such a case, the summing calculation amount has been signif cantly reduced by applying the expressions that use, in each iteration, the respective values obtained by processing samples of a previously downsampled realization and some samples of the currently downsampled one. We formulate def nitions and prove the corollaries that refer to the recursive Fourier coeff cient calculation and present here an example. Finally, the simulation results for the bandlimited signal with a triangularshaped spectrum are presented.

Key words: Digital signal processing (DSP); discrete Fourier transform (DFT); discrete-time signals; realization; decimation; f ltering; downsampling; recognition.

1. Introduction

The sampling operation of an analogous signal $U(t)$ with a sampling frequency F_s signif cantly higher than twice the highest signal's frequency B is frequently used while processing $U(t)$ digitally [1, 2, 9, 13, 14]. There are some reasons for performing such an oversampling. One of the main ones of them is a less complex and inexpensive anti-aliasing f lter: a signal can be f ltered digitally, and afterwards, downsampled to the desired sampling frequency by reducing a large digital data set considerably [6, 8]. A time-scaling operation is used here that is equivalent to changing the sampling rate of an analogous signal $U(t)$ from $1/T_s$ to $1/PT_s$, where T_s is a sampling period and its reciprocal $1/T_s = F_s$, i.e., decreasing the sampling rate by factor P. On the other hand, the number of samples N to be processed decreases P times, as well. In general, the basic sampling frequency F_s could be decreased by varying the integer number of times if not the fact that the data decimation process ought to be f nished before the frequency content of the downsampled signal is above the new Nyquist frequency [13]. In the opposite case, the spectrum $X(F/F_s)$ of the discrete-time signal $U(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}-1}$ contains aliased frequency components F of the spectrum $X_a(F)$ of an analogous signal $U(t)$, because the downsampling in the frequency domain leads to the spread of signal's spectrum by the same factor P [3, 4]. That is why there arises a question: how much can we downsample the signal in order to reduce the amount of samples to be processed, without loss of information. Therefore, to prevent the loss of information due to aliasing of frequencies, it is important to choose a proper criterion and to develop a technique that could be used to recognize non-aliased realizations in the given set of realizations obtained after multifold downsampling of an oversampled discrete-time bandlimited signal. There also arises a problem to reduce the number of routine mathematical operations used to recognize the non-aliased realization. How to reduce the number of summing operations while calculating the first and the second order statistical moments of decimated realizations using recursive equations is shown in [10, 11]. Note that recursive or iterative schemes are frequently used to reduce the computational complexity [5, 7]. The criterion for recursive stopping of multifold downsampling of discrete-time bandlimited signals has been developed in [12]. On the other hand, in this paper the general problem is solved – the data decimation is used by means of filtering and downsampling operations.

In Section 2, the statement of the problem is given. In Section 3, the criterion based on the calculation of the Fourier series coefficient located at Nyquist frequency has been proposed. Recursive expressions for recognition of the very last non-aliased decimated realization in the given set of downsampled realizations are made up in Section 4. In Section 5, an example is given for any realization of 128 samples. The simulation results are presented in Section 6. Section 7 contains conclusions.

2. Statement of the problem

We consider a discrete-time bandlimited signal $U(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}-1}$ that is obtained by uniform sampling with the sampling frequency F_s its continuous-time counterpart $U(t)$ having a bandwidth [–B, B]. Here N is the general number of samples of the basic signal $U(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}-1}$ under consideration.

Let us assume that, after onetime lowpass filtering, used to prevent the aliasing of frequencies, and after following multifold downsampling of the realization $u(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}-1}$ of the basic signal $U(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}-1}$, one has to store in the memory of a computer the set $\mathbf{\Omega}$ of realizations:

$$\begin{aligned} x_1(k) &\equiv u(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}-1}, \qquad (1)\\ x_2(k) &\equiv u(k2T_s)\ \forall k \in \overline{0, \mathrm{N}/2-1},\\ x_3(k) &\equiv u(k2^2T_s)\ \forall\ k \in \overline{0, \mathrm{N}/2^2-1},\\ &\vdots\\ x_{m-1}(k) &\equiv u(k2^{(m-1)}T_s)\ \forall k \in \overline{0, \mathrm{N}/2^{(m-2)}-1},\\ x_m(k) &\equiv u(k2^{m}T_s)\ \forall\ k \in \overline{0, \mathrm{N}/2^{(m-1)}-1}. \end{aligned}$$

In spite of lowpass filtering of the realization $u(kT_s)$ $\forall\ k \in \overline{0, \mathrm{N}-1}$ the maximal frequencies of some realizations, obtained after repeated downsampling, could be higher than the varying new Nyquist frequencies leading to overlapping of respective signal frequencies when the spectrum replicates. Thus, the set $\mathbf{\Omega}$ could be subdivided in turn, into two subsets: a subset of non-aliased realizations $\mathbf{\Omega}_1$, and a subset of aliased ones $\mathbf{\Omega}_2$. The last realization of subset $\mathbf{\Omega}_1$ is the very last non-aliased realization, after which there follows the first aliased one from the subset $\mathbf{\Omega}_2$.

The aim of the given paper is, firstly, to choose some criterion that could be used to recognize the very last downsampled realization in the given set $\mathbf{\Omega}$ with the maximal frequency that is still below the new Nyquist frequency, and, secondly, to considerably reduce the number of routine calculations required to recognize the same non-aliased realization.

3. Determination of the criterion

For the initial non-decimated centered realization $x_1(k) \equiv u(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}}$ of a discrete-time signal $U(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}}$, one can write the Fourier series expansion [1]

$$x_1(k) = A_1(0) + \sum_{q=1}^{2^{(n-1)}} A_1(q)\cos(2^{-(n-1)}\pi qk) \qquad (2)$$

$$+ \sum_{q=1}^{2^{(n-1)}-1} B_1(q)\sin(2^{-(n-1)}\pi qk), \forall\ k \in \overline{0, 2^{\mathrm{n}}-1}.$$

Here N is the period of the same discrete-time signal, divisible n times by 2, i.e., $\mathrm{N} = 2^n$; and

$$A_1(0) = \frac{1}{2^n}\sum_{k=0}^{2^{\mathrm{n}}-1} x_1(k) = 0, \qquad (3)$$

$$A_1(q) = 2^{-(n-1)}\sum_{k=0}^{2^{\mathrm{n}}-1} x_1(k)\cos(2^{-(n-1)}\pi qk), \qquad (4)$$

$$A_1(2^{(n-1)}) = \frac{1}{2^n}\sum_{k=0}^{2^{\mathrm{n}}-1} x_1(k)\cos\pi k, \qquad (5)$$

$$B_1(q) = 2^{-(n-1)}\sum_{k=0}^{2^{\mathrm{n}}-1} x_1(k)\sin(2^{-(n-1)}\pi qk), \qquad (6)$$

$$q = 1, 2, ..., 2^{(n-1)}-1$$

are the respective coefficients of real DFT.

It is known that, for a discrete-time bandlimited signal, Fourier series coefficients are nonzero inside the band [–B, B], and zero outside the same band [13]. Therefore, the values of coeff cients appearing far from the zero value for frequencies outside the bandwidth of some decimated realization could show us that it is time to f nish downsampling. Note that calculations could be signif cantly reduced if for each realization $x_i \; \forall \; i \in \overline{1,m}$ from the set $\mathbf{\Omega}$, only the values of $A_i(2^{(n-i)}) \; \forall \, i \in \overline{1,m}$ were calculated, according to the expression

$$A_i(2^{(n-i)}) = \frac{1}{2^{n-i+1}} \sum_{k=0}^{2^{n-i+1}-1} x_i(k) \cos \pi k, \quad (7)$$

$\forall \; i \in \overline{1,m}$, especially when the bandwidth B is known only approximately.

Let us now formulate statements that determine the criterion for recognizing non-aliased decimated realization from the set $\mathbf{\Omega}$.

Proposition 1. *If the shifted replicas of the spectrum $X_a(F)$ of an analogous signal $U(t)$, that are available in the spectrum $X_i(e^{j\omega})$ of any non-decimated and decimated realizations $x_i(k) \equiv u(kT_s)$ $\forall \; k \in \overline{0, 2^{n-i+1}-1}$, and $\forall \; i \in \overline{1,m}$, i.e.*

$$X_i(e^{j\omega}) = \frac{1}{2^i T_s} \sum_{\nu=-\infty}^{\infty} X_a\left(\frac{j\omega}{2^i T_s} - \frac{j2\pi\nu}{2^i T_s}\right), \quad (8)$$

do not overlap, then the value of the coefficient $A_i(2^{(n-i)}) \; \forall \, i \in \overline{1,m}$, located at $F_s/2$, is zero for any i-th decimated realization from the set $\mathbf{\Omega}$. The value of $A_i(2^{(n-i)}) \; \forall \, i \in \overline{1,m}$ is non-zero, on the contrary.

Remark 1. Aliasing of frequencies in any decimated realization from the set $\mathbf{\Omega}$ is absent, if and only if $A_i(2^{(n-i)}) = 0 \; \forall \; i \in \overline{1,m}$, and it is present, otherwise.

Proposition 2. *If, for each realization from the set $\mathbf{\Omega}$, the values of the coefficient $A_i(2^{(n-i)}) \; \forall \; i \in \overline{1,m}$: $A_1(2^{(n-1)}), A_2(2^{(n-2)}), A_3(2^{(n-3)}), \ldots,$ $A_m(2^{(n-m)})$ are calculated, then it is easy to recognize the very last non-aliased s-th realization: its value $A_s(2^{(n-s)})$ is zero, while the value of the very first aliased s+1-st realization $A_{s+1}(2^{(n-s-1)})$ is non-zero.*

Remark 2. The values $A_i(2^{(n-i)}) \; \forall \; i \in \overline{1,m}$ could be used to recognize non-aliased realizations. In general, it is important to determine the very last non-aliased realization that divides the set $\mathbf{\Omega}$ into subsets $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$.

Note that, really, the abovementioned coeff cient values will be seldom equal to zero even for non-aliased realizations due to the f nite number of samples in a respective realization and, especially, because of additive noise that has been added to samples of the discrete-time signal and its downsampled versions. Therefore, let us choose the value of the form

$$c_i = \frac{\| A_i(2^{(n-i)}) - A_{i+1}(2^{(n-i-1)}) \|_E^2}{\| A_{i+1}(2^{(n-i-1)}) \|_E^2} 100\%, \quad (9)$$

$\forall \; i \in \overline{1, m-1}$, as the criterion of recognition of the very last non-aliased realization assuming that the values of $A_i(2^{(n-i)})$, $A_{i+1}(2^{(n-i-1)})$ will be calculated in reverse order, i.e., f rst, we calculate $A_{i+1}(2^{(n-i-1)})$, afterwards, $A_i(2^{(n-i)})$. The gross value of (9) as compared with its other values shows us that the current realization is the last non-aliased one.

By continuing the procedure in reverse order, one could obtain the recursive formulas for calculating $A_i(2^{(n-i)}) \; \forall \, i \in \overline{1,m}$.

4. Recursive expressions

In order to reduce calculations completely, we will work out a recursive expression to be used to determine $A_i(2^{(n-i)}) \; \forall \; i \in \overline{1,m}$. Firstly, let us formulate corollaries on their calculation, assuming, for simplicity, that the basic T_s for each downsampled realization is increased by 2 times. In general, it can be increased arbitrarily integer number times until the maximal frequency of the decimated realization is still below the new Nyquist frequency.

Corollary 1. *The value of the coefficient $A_i(2^{(n-i)})$ of each i-th realization from the set $\mathbf{\Omega}$ is calculated in reverse order, using the recursive expression of the form*

$$A_i(2^{(n-i)}) = \frac{1}{2^{n-i+1}} \{ 2^{(n-i)} A_{i+1}(2^{(n-i-1)}) + 2 \sum_{k=1}^{2^{(n-i-1)}} x_i(2^2 k - 2) - \sum_{k=1}^{2^{(n-i)}} x_i(2k-1) \} \quad (10)$$
$$\forall \; i \in \overline{1, m-1}.$$

Here $A_i(2^{(n-i)})$, $A_{i+1}(2^{(n-i-1)})$ are current and previous values of the Fourier coefficient that has been located at Nyquist frequency, $x_i(k)$ $\forall \; i \in \overline{1, m-1}$ is a current realization from the set Ω.

Proof. For the initial non-decimated realization $x_1(k)$ of a discrete-time signal $U(kT_s) \; \forall \; k \in \overline{0, 2^n}$, the

Fourier coeff cient

$$A_1(2^{(n-1)}) = \frac{1}{2^n}\sum_{k=0}^{2^n-1} x_1(k)\cos\pi k \quad (11)$$
$$= \frac{1}{2^n}\sum_{k=0}^{2^n-1}(-1)^k x_1(k)$$
$$= \frac{1}{2^n}\{\sum_{k=0}^{2^{(n-1)}-1} x_1(2k) - \sum_{k=1}^{2^{(n-1)}} x_1(2k-1)\},$$

located at $F_s/2$, which corresponds to the normalized frequency π, could be calculated by

$$A_1(2^{(n-1)}) = \frac{1}{2^n}\sum_{k=0}^{2^n-1}(-1)^k x_1(k) \quad (12)$$
$$= \frac{1}{2^n}\{\sum_{k=0}^{2^0} x_1(2^{(n-1)}k) + \sum_{k=1}^{2^1} x_1(2^{(n-1)}k - 2^{(n-2)})$$
$$+\sum_{k=1}^{2^2} x_1(2^{(n-2)}k - 2^{(n-3)}) + \ldots + \sum_{k=1}^{2^{(n-3)}} x_1(2^3k - 2^2)$$
$$+ \sum_{k=1}^{2^{(n-2)}} x_1(2^2k - 2) - \sum_{k=1}^{2^{(n-1)}} x_1(2k-1)\},$$

under the assumption that, in curly braces of (12), there exist n sums in all, and the last sum has a negative sign. Then, for the very f rst downsampled realization $x_2(k)$, the coeff cient $A_2(2^{(n-2)})$ is of the form

$$A_2(2^{(n-2)}) = \frac{1}{2^{(n-1)}}\sum_{k=0}^{2^{(n-1)}-1}(-1)^k x_2(k) \quad (13)$$
$$= \frac{1}{2^{(n-1)}}\{\sum_{k=0}^{2^0} x_2(2^{(n-2)}k) + \sum_{k=1}^{2^1} x_2(2^{(n-2)}k - 2^{(n-3)})$$
$$+\sum_{k=1}^{2^2} x_2(2^{(n-3)}k - 2^{(n-4)}) + \ldots + \sum_{k=1}^{2^{(n-4)}} x_2(2^3k - 2^2)$$
$$+ \sum_{k=1}^{2^{(n-3)}} x_2(2^2k - 2) - \sum_{k=1}^{2^{(n-2)}} x_2(2k-1)\},$$

under the assumption that, there exist $n-1$ sums, in curly braces of (13), and the last sum has a negative sign, too. Continuing the procedure, f nally one could obtain the formula for calculating $A_{(m-1)}(2^{(n-m)})$ by means of formulas below

$$A_{(m-1)}(2^{(n-m)}) = \frac{1}{2^{(n-m+1)}}\sum_{k=0}^{2^{(n-m+1)}-1}(-1)^k x_{(m-1)}(k)$$
$$= \frac{1}{2^{(n-m+1)}}\{\sum_{k=0}^{2^0} x_{(m-1)}(2^{(n-m)}k)$$
$$+\sum_{k=1}^{2^1} x_{(m-1)}(2^{(n-m)}k - 2^{(n-m-1)})$$
$$+\sum_{k=1}^{2^2} x_{(m-1)}(2^{(n-m-1)}k - 2^{(n-m-2)})$$
$$+\ldots+\sum_{k=1}^{2^{(n-m-2)}} x_{(m-1)}(2^3k - 2^2)$$
$$+\sum_{k=1}^{2^{(n-m-1)}} x_{(m-1)}(2^2k - 2)$$
$$-\sum_{k=1}^{2^{(n-m)}} x_{(m-1)}(2k-1)\},$$

and for calculating $A_m(2^{(n-m-1)})$ according to the expression

$$A_m(2^{(n-m-1)}) = \frac{1}{2^{(n-m)}}\sum_{k=0}^{2^{(n-m)}-1}(-1)^k x_m(k)$$
$$= \frac{1}{2^{(n-m)}}\{\sum_{k=0}^{2^0} x_m(2^{(n-m-1)}k)$$
$$+\sum_{k=1}^{2^1} x_m(2^{(n-m-1)}k - 2^{(n-m-2)})$$
$$+\sum_{k=1}^{2^2} x_m(2^{(n-m-2)}k - 2^{(n-m-3)}) + \ldots$$
$$+\sum_{k=1}^{2^{(n-m-3)}} x_m(2^3k - 2^2)$$
$$+\sum_{k=1}^{2^{(n-m-2)}} x_m(2^2k - 2)$$
$$-\sum_{k=1}^{2^{(n-m-1)}} x_m(2k-1)\},$$

The next-to-last and last expressions have $n-m+1$ and $n-m$ sums, respectively. In both expressions the last sums have negative signs.

One can rewrite (12) as follows:

$$A_1(2^{(n-1)}) = \frac{1}{2^n}\{2^{(n-1)}A_2(2^{(n-2)}) \quad (14)$$
$$+2\sum_{k=1}^{2^{(n-2)}} x_1(2^2k-2) - \sum_{k=1}^{2^{(n-1)}} x_1(2k-1)\},$$

having in mind that $x_2(0) = u(0), x_2(1) = u(2T_s)$, $x_2(2) = u(4T_s), ..., x_2(2^{(n-1)}-2) = u(2^nT_s - 4T_s), x_2(2^{(n-1)}-1) = u(2^nT_s - 2T_s)$, and that $x_2(0) \equiv x_1(0), x_2(1) \equiv x_1(2), x_2(2) \equiv x_1(4), ...,$ $x_2(2^{(n-1)}-2) \equiv x_1(2^n-4), x_2(2^{(n-1)}-1) \equiv x_1(2^n-2)$.

Equation (13) can also be rewritten in a recursive form:

$$A_2(2^{(n-2)}) = \frac{1}{2^{(n-1)}}\{2^{(n-2)}A_3(2^{(n-3)})$$
$$+2\sum_{k=1}^{2^{(n-3)}} x_2(2^2k-2) - \sum_{k=1}^{2^{(n-2)}} x_2(2k-1)\},$$

if we are aware that $x_3(0) = u(0), x_3(1) = u(4T_s), x_3(2) = u(8T_s), ..., x_3(2^{(n-2)}-2) = u(2^nT_s - 8T_s), x_3(2^{(n-2)}-1) = u(2^nT_s - 4T_s)$, and that $x_3(0) \equiv x_1(0), x_3(1) \equiv x_1(4), x_3(2) \equiv x_1(8), ..., x_3(2^{(n-2)}-2) \equiv x_1(2^n-8), x_3(2^{(n-2)}-1) \equiv x_1(2^n-4)$.

Proceeding with the procedure, for $A_{(m-2)} \equiv A_{(m-2)}(2^{(n-m+2)})$, $A_{(m-1)} \equiv A_{(m-1)}(2^{(n-m+1)})$, and $A_m \equiv A_m(2^{(n-m)})$, one could get recursive formulas of the form

$$A_{(m-2)} = \frac{1}{2^{n-m+3}}\{2^{(n-m+2)}A_{(m-1)} \quad (15)$$
$$+2\sum_{k=1}^{2^{(n-m+1)}} x_{(m-2)}(2^2k-2) - \sum_{k=1}^{2^{(n-m+2)}} x_{(m-2)}(2k-1)\},$$

and

$$A_{(m-1)} = \frac{1}{2^{(n-m+2)}}\{2^{(n-m+1)}A_m \quad (16)$$
$$+2\sum_{k=1}^{2^{(n-m)}} x_{m-1}(2^2k-2) - \sum_{k=1}^{2^{(n-m+1)}} x_{m-1}(2k-1)\},$$

respectively. Thus, the general expression for calculating $A_i(2^{(n-i)}) \ \forall \ i \in \overline{1, m-1}$ is of the form (10). □

Corollary 2. *The values of $A_{(i+1)}(2^{(n-i-1)})$ of each i-th realization from the set $\mathbf{\Omega}$ are calculated in direct order using the recursive expression of the form*

$$A_{i+1}(2^{(n-i-1)}) = 2A_i(2^{(n-i)}) \quad (17)$$
$$-\frac{1}{2^{(n-i-1)}}\sum_{k=1}^{2^{(n-i-1)}} x_i(2^2k-2)$$
$$+\frac{1}{2^{(n-i)}}\sum_{k=1}^{2^{(n-i)}} x_i(2k-1) \forall \ i \in \overline{1, m-1}.$$

Proof. The proof of Corollary 2 is similar to that of Corollary 1. □

Remark 3. Recursive expression for reverse order calculations according to formula (10) allows us to decrease the number of summing operations as compared with the ordinary expression (7) or even with the direct order recursive expression of the form (17).

5. Example

After decimating the basic realization $x_1(k) \equiv u(kT_s) \ \forall k \in \overline{0, 2^7-1}$, we get the set $\mathbf{\Omega}$ of the next realizations:

$$x_1(k) \equiv u(kT_s) \ \forall k \in \overline{0, 2^7-1}, \quad (18)$$
$$x_2(k) \equiv u(k2T_s) \ \forall k \in \overline{0, 2^6-1},$$
$$x_3(k) \equiv u(k4T_s) \ \forall \ k \in \overline{0, 2^5-1},$$
$$x_4(k) \equiv u(k8T_s) \ \forall \ k \in \overline{0, 2^4-1},$$
$$x_5(k) \equiv u(k16T_s) \ \forall \ k \in \overline{0, 2^3-1}.$$

We rewrite the value $A_1(2^6)$ of the non-decimated realization $x_1(k) \ \forall \ k \in \overline{0, 2^7-1}$, as follows:

$$A_1(2^6) = \frac{1}{2^7}\sum_{k=0}^{2^7-1}(-1)^k x_1(k) = \frac{1}{2^7}\{\sum_{k=0}^{2^6-1} x_1(2k) \quad (19)$$
$$-\sum_{k=1}^{2^6} x_1(2k-1)\} = \frac{1}{2^7}\{\sum_{k=0}^{2^0} x_1(2^6k)$$
$$+\sum_{k=1}^{2^1} x_1(2^6k-2^5) + \sum_{k=1}^{2^2} x_1(2^5k-2^4)$$
$$+\sum_{k=1}^{2^3} x_1(2^4k-2^3) + \sum_{k=1}^{2^4} x_1(2^3k-2^2)$$
$$+\sum_{k=1}^{2^5} x_1(2^2k-2) - \sum_{k=1}^{2^6} x_1(2k-1)\}.$$

Afterwards, the values $A_2(2^5), A_3(2^4), A_4(2^3)$ have been found by the formulas:

$$A_2(2^5) = \frac{1}{2^6}\{\sum_{k=0}^{2^0} x_2(2^5k) + \sum_{k=1}^{2^1} x_2(2^5k - 2^4) + \sum_{k=1}^{2^2} x_2(2^4k - 2^3) + \sum_{k=1}^{2^3} x_2(2^3k - 2^2) + \sum_{k=1}^{2^4} x_2(2^2k - 2) - \sum_{k=1}^{2^5} x_2(2k - 1)\}, \quad (20)$$

$$A_3(2^4) = \frac{1}{2^5}\{\sum_{k=0}^{2^0} x_3(2^4k) + \sum_{k=1}^{2^1} x_3(2^4k - 2^3) + \sum_{k=1}^{2^2} x_3(2^3k - 2^2) + \sum_{k=1}^{2^3} x_3(2^2k - 2) - \sum_{k=1}^{2^4} x_3(2k - 1)\}, \quad (21)$$

$$A_4(2^3) = \frac{1}{2^4}\{\sum_{k=0}^{2^0} x_4(2^3k) + \sum_{k=1}^{2^1} x_4(2^3k - 2^2) + \sum_{k=1}^{2^2} x_4(2^2k - 2) - \sum_{k=1}^{2^3} x_4(2k - 1)\}, \quad (22)$$

and

$$A_5(2^2) = \frac{1}{2^3}\{\sum_{k=0}^{2^0} x_5(2^2k) + \sum_{k=1}^{2^1} x_5(2^2k - 2) - \sum_{k=1}^{2^2} x_5(2k - 1)\}, \quad (23)$$

respectively. Proceeding with calculations in reverse order, one could find that the first and second terms in curly braces of (23) are equal to the first and second terms of equation (22), respectively. The absolute value of the third term in (23) is equal to the same value of the third term in expression (22). Thus, one can write

$$A_4(2^3) = \frac{1}{2^4}\{2^3 A_5(2^2) + 2\sum_{k=1}^{2^2} x_4(2^2k - 2) - \sum_{k=1}^{2^3} x_4(2k - 1)\}. \quad (24)$$

Continuing the process in the same order, one could discover that the first, second, and third terms in curly braces of (22) are equal to the respective terms in (21). Note that the fourth terms in the same formulas are written with different signs. Therefore, one can obtain the recursive formula

$$A_3(2^4) = \frac{1}{2^5}\{2^4 A_4(2^3) + 2\sum_{k=1}^{2^3} x_3(2^2k - 2) - \sum_{k=1}^{2^4} x_3(2k - 1)\}. \quad (25)$$

Finally, by comparing the respective terms of the corresponding equations that have been left, we have

$$A_2(2^5) = \frac{1}{2^6}\{2^5 A_3(2^4) + 2\sum_{k=1}^{2^4} x_2(2^2k - 2) - \sum_{k=1}^{2^5} x_2(2k - 1)\}, \quad (26)$$

and

$$A_1(2^6) = \frac{1}{2^7}\{2^6 A_2(2^5) + 2\sum_{k=1}^{2^5} x_1(2^2k - 2) - \sum_{k=1}^{2^6} x_1(2k - 1)\}. \quad (27)$$

Thus, one can determine $A_4(2^3), A_3(2^4), A_2(2^5), A_1(2^6)$ in reverse order and recursively, beginning with (24), and finishing with (27). Afterwards, three values of the recognition criterion

$$c_i = \frac{\| A_i(2^{(7-i)}) - A_{i+1}(2^{(7-i-1)}) \|_E^2}{\| A_{i+1}(2^{(7-i-1)}) \|_E^2} 100\%, \quad (28)$$

$\forall\ i \in \overline{1,3}$ are calculated in reverse order, too. The gross meaning of (28) as compared with its other meanings shows us that the current realization is the last non-aliased one. Note that $A_1(2^6), A_2(2^5), A_3(2^4)$,

$A_4(2^3)$ can be calculated by the ordinary formula

$$A_i(2^{(7-i)}) = \frac{1}{2^{8-i}} \sum_{k=0}^{2^{8-i}-1} (-1)^k x_i(k) \quad (29)$$

for $i \in \overline{1,4}$. We shall need $2^7 + 2^6 + 2^5 + 2^4$ summing operations in total for their calculation. Recursive calculations according to formulas (24) – (27) allow us to decrease the number of summing operations as compared with the ordinary expression (29). In such an example, while calculating $A_4(2^3), A_3(2^4), A_2(2^5)$, $A_1(2^6)$, we avoid 4, 8, 16, and 32 summing operations, respectively. In general, if we have some realization consisting of 128 samples we need 60 less summing operations while calculating the respective Fourier coeff cient values in comparison with the operations performed using the ordinary formula (29). On the other hand, in such an example, there appear several additional multiplication operations in each recursive iteration as in [10, 11].

6. Simulation results

It is emphasized in [8] that, in a theoretical discussion of sampling theory, it is usual to represent the signal of interest with a triangularshaped Fourier spectrum. A triangle can be obtained in the frequency domain in view that [8]

$$\frac{sin(\omega_c n/2)}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod(\frac{\omega-\theta}{\omega_c}) \prod(\frac{\theta}{\omega_c}) d\theta. (30)$$

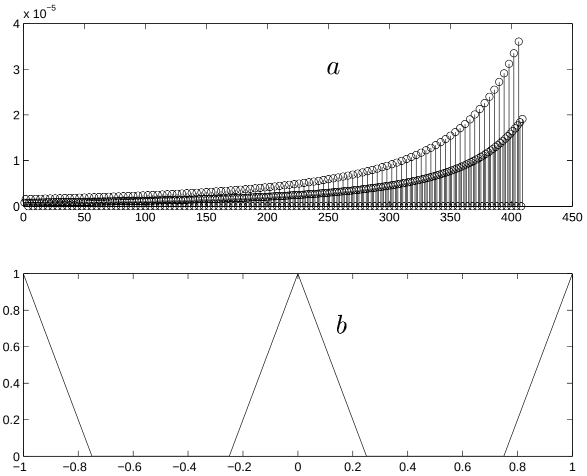

Fig. 1. A short segment of the simulated initial realization (33) to be processed (a), and its unit height spectrum (b)

Here $\frac{\omega_c}{2\pi} = B$. It follows then that, for a unit height spectrum, we have the transform pair [8]

$$\frac{2\pi}{\omega_c}\left(\frac{sin(\omega_c n/2)}{\pi n}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow} \Lambda(\frac{\omega}{\omega_c}). \quad (31)$$

Here $\Lambda(\frac{\omega}{\omega_c})$ is a unit height spectrum with the bandwidth $[-\omega_c, \omega_c]$. Using the sinc() function in MATLAB, which is def ned as [8]

$$sinc(x) \equiv \frac{sin\pi x}{\pi x} \quad (32)$$

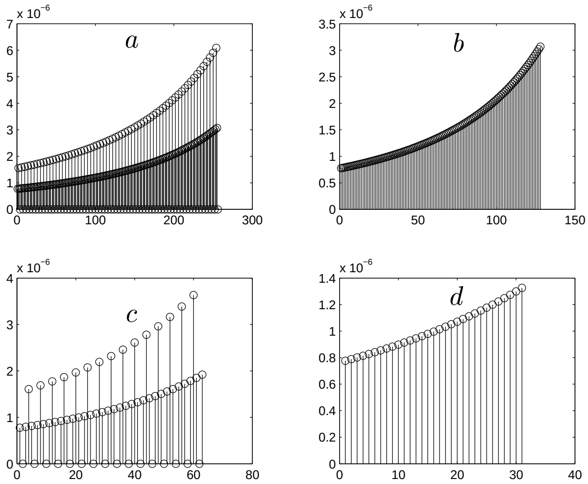

Fig. 2. A set of realizations: initial (33), (a) initial realization downsampled by 2 (b), by 3 (c), and by 4 samples (d)

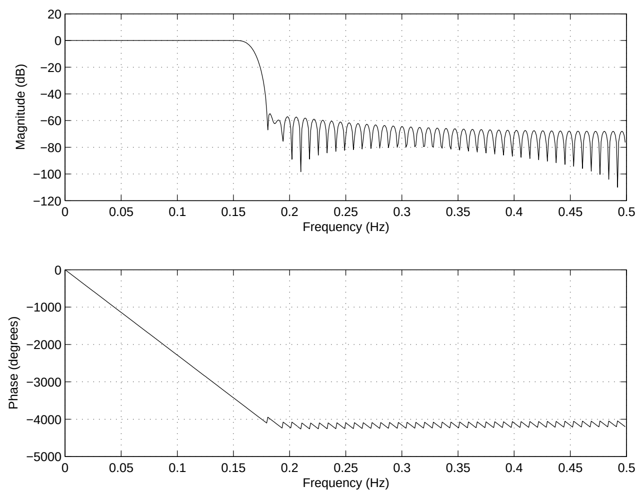

Fig. 3. The magnitude and the phase of a lowpass digital FIR f lter with 128 coeff cients and cutoff frequency $\pi/3$ [8]

the signal $U(kT_s)\ \forall\ k \in \overline{0, \mathrm{N}-1}$, that for $T_s = 1$ is of the form

$$U(k) = \frac{1}{4} sinc(\frac{1}{4}(k - 512))^2 \qquad (33)$$

$\forall\ k \in \overline{0, 1023}$, has been generated. The signal to be decimated and its spectrum before decimation are presented in Fig1a and Fig1b, respectively. Afterwards, two experiments are carried out. In both experiments the realization of signal (33) has been downsampled in three different ways: by 2, 4, 8, 16 samples carrying out the first experiment, and by 3, 6 and 9 samples during the second one. Then, the initial realization of signal (33), firstly, was filtered by a digital FIR filter (see Fig. 3) and, secondly, downsampled in the same ways, too [2, 8]. The different typical non-filtered and filtered realizations of the initial signal realization (33) (Fig. 1a) chosen after carrying out both experiments are shown in Fig's. 2, 4. Spectra for the non-filtered realizations are shown in Fig.5, while for the filtered ones in Fig. 6. Note that Fig. 2a, b, c, d corresponds to Fig. 5a, b, c, d, and Fig. 4a, b, c, d – to 6a, b, c, d, respectively. From the simulation results (Fig's. 5c, 5d) it follows that, with a decrease in the sampling rate by P more than 2 there appears aliasing of frequencies for non-filtered realizations. On the other hand, the aliasing for filtered realizations is present only if P is more than 3 (Fig.5d). In such a case, the Fourier coefficient located at Nyquist frequency is already unequal to zero. The values of the recognition criterion have been calculated in reverse order, as follows:

$$c_4 = \frac{\| A_4(2^{(6)}) - A_5(2^{(5)}) \|_E^2}{\| A_5(2^{(5)}) \|_E^2} 100\% = 26.05\%, \qquad (34)$$

$$c_3 = \frac{\| A_3(2^{(7)}) - A_4(2^{(6)}) \|_E^2}{\| A_4(2^{(6)}) \|_E^2} 100\% = 30.43\%,$$

$$c_2 = \frac{\| A_2(2^{(8)}) - A_3(2^{(7)}) \|_E^2}{\| A_3(2^{(7)}) \|_E^2} 100\% = 100\%,$$

$$c_1 = \frac{\| A_1(2^{(9)}) - A_2(2^{(8)}) \|_E^2}{\| A_2(2^{(8)}) \|_E^2} 100\% = 8.96e^{+022}\%,$$

$\forall\ i \in \overline{1, 4}$.

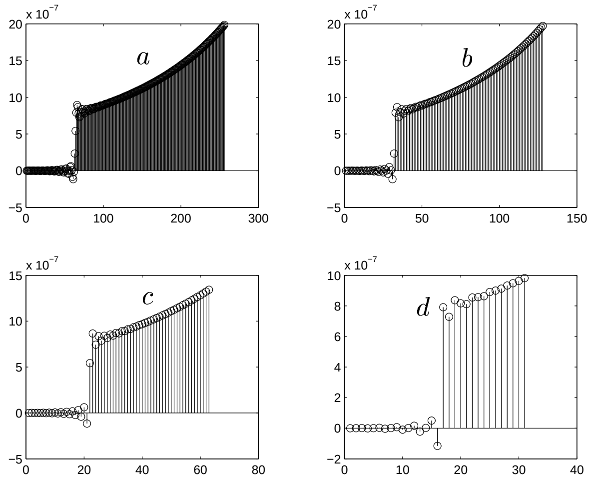

Fig. 4. A set of filtered realizations: initial (33), filtered with the digital FIR filter (a), initial filtered realization downsampled by 2 (b), by 3 (c), and by 4 samples (d)

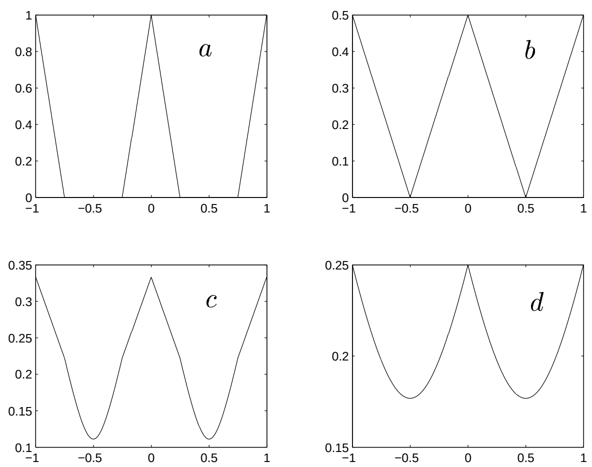

Fig. 5. Spectra of the initial realization (a), and downsampled but non-filtered versions (b, c, d)

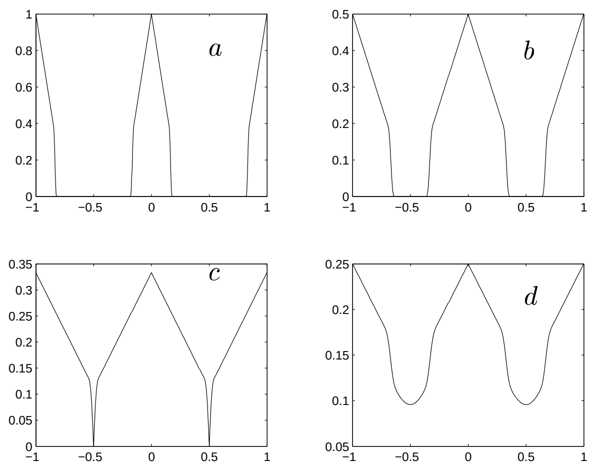

Fig. 6. Spectra of filtered realizations: initial without downsampling (a), and its filtered and downsampled versions (b, c, d)

Here

$$A_i(2^{(10-i)}) = \frac{1}{2^{11-i}} \sum_{k=0}^{2^{11-i}-1} (-1)^k x_i(k) \qquad (35)$$

$\forall\, i \in \overline{1,5}$,

$$\begin{aligned} x_1(k) &\equiv u(k)\ \forall\ \ k \in \overline{0, 2^{10}-1}, \qquad (36)\\ x_2(k) &\equiv u(2k)\ \forall k \in \overline{0, 2^{9}-1},\\ x_3(k) &\equiv u(2^2k)\ \forall\ k \in \overline{0, 2^{8}-1},\\ x_4(k) &\equiv u(2^3k)\ \forall\ k \in \overline{0, 2^{7}-1},\\ x_5(k) &\equiv u(2^4k)\ \forall\ k \in \overline{0, 2^{6}-1}. \end{aligned}$$

The Fourier coeff cient value obtained for the realization $x_1(k) \equiv u(k)\ \forall\ \ k \in \overline{0, 2^{10}-1}$, downsampled by 3 samples, was chosen from the second experiment in order to calculate the criterion. We have def ned its value 28.04%. Then the values c_1, c_2, c_3, c_4 were calculated after f ltering the f rst realization from the set of realizations (36). We have obtained in reverse order the following values: $c_4 = 0.01\%, c_3 = 14.92\%, c_2 = 99.99\%, c_1 = 2.70\%$. The Fourier coeff cient value calculated for the f ltered realization $x_1(k) \equiv u(k)\ \forall\ \ k \in \overline{0, 2^{10}-1}$, downsampled by 3 samples, was chosen from the second experiment. The value of the criterion, based on the abovementioned Fourier coeff cient, is 99.99%, too. Thus, after f ltering we can downsample the f ltered realization $x_1(k) \equiv u(k)\ \forall\ \ k \in \overline{0, 2^{10}-1}$ by 3 samples, while non-f ltered realization only by 2 samples, avoiding the aliasing of frequencies. It could be emphasized here that only to calculate the values of (35), $2^{10} + 2^9 + 2^8 + 2^7 + 2^6$ summing operations will be needed: in total, 1984 summing operations. Recursive calculations in reverse order according to formula (10) allow us to decrease the number of summing operations considerably. Note that, in such a case, because of reverse order calculations, it is enough for us to determine only the values of $A_5(2^5), A_4(2^6), A_3(2^7), A_2(2^8)$. Thus, we spend 2^6 summing operations for $A_5(2^5)$, 97 for $A_4(2^6)$, 193 for $A_3(2^7)$, and 385 for $A_2(2^8)$, respectively, in total, 739 summing and 6 multiplication operations.

7. Conclusions

While processing discrete-time signals, there arises a problem to retrieve maximal information as well as to reduce the amount of calculations on the basis of samples, especially, if oversampled signals are available. In such a case, the data decimation is used by means of f ltering and downsampling operations. However, it is unknown beforehand how much we can downsample a signal in order to reduce the amount of samples to be processed, without loss of information due to aliasing of frequencies. For multifold downsampled bandlimited signals the discrete-time Fourier series coeff cient values located at Nyquist frequency have been proposed, to determine each signal realization. The number of operations for its speedy calculating is essentially reduced using original recursive expression (10) for reverse order calculations. The criterion of recognition (9) of the very last non-aliased realization, based on the calculation of the abovementioned Fourier coeff cient values, has been established. The simulation results for the bandlimited signal with a triangularshaped spectrum (see Fig's. 1–6) have shown us the eff ciency of the recursive approach for recognition of the subset of non-aliased downsampled realizations in the given set of realizations.

References

[1] **J. S. Bendat, A. G. Piersol.** Measurement and Analysis of Random Data. *John Wiley Sons, inc., N.Y.* 1967.

[2] **R. E. Crochiere, L. R. Rabiner.** Optimum FIR digital f lter implementations for decimations, interpolation and narrowband f ltering. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, Oct. 1975, ASSP-23(5), 444 - 456.

[3] **R. E. Crochiere, L. R. Rabiner.** Further considerations in the design of decimators and interpolators. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, August 1976, ASSP-24(4), 296 - 311.

[4] **R. E. Crochiere, L. R. Rabiner.** Interpolation and decimations of digital signals - a tutorial review. *IEEE Proceedings*, March 1981, 69(3), 300 - 331.

[5] **B. Goosens, J. Aelterman, A. Pižurica, W. Philips.** A recursive scheme for computing autocorrelation functions of decimated complex wavelet sub-bands. *IEEE Transactions on Signal Processing*, July 2010, 58(7), 3907 - 3912.

[6] **L. Hars.** How to decimate with a DSP. *DSP World, Conference Proceedings, Orlando, 1999*. http://www.hars.us/.

[7] **A. Misevičius, D. Rubliauskas, V. Barkauskas.** Some further experiments with the genetic algorithm for the quadratic assignment problem. *Information Technology and Control*,2009, 38(4), 325 - 332.

[8] Multirate sampling simulation using MATLAB's signal processing toolbox. Introduction. ECE5650/4650. http://www.eas.uccs.edu/wickert/ece5650/notes/sampling-theory/multirate-sim.pdf.

[9] **M. C. Ndinechi, N. Onwuchekwa, G. A. Chukwudebe.** Algorithm for applying decimator structures in digital signal processing systems for energy conversation. *International Journal of Soft Computing,*2009, 4(6), 236 - 242.

[10] **R. Pupeikis.** On recursive expressions for statistics of decimated sequences. *Lietuvos matematikos rinkinys, spec.nr.,* 2007, T. 47, 411-416.

[11] **R. Pupeikis.** On the processing of decimated signals. *Informatica,* 2008, 19(4), 617-630.

[12] **R. Pupeikis.** On recursive stopping of decimation of discrete-time bandlimited signals. *Lietuvos matematikos rinkinys,* 2011, T. 52, 338-342.

[13] **J. G. Proakis, D. G. Manolakis.** Digital Signal Processing, Principles, Algorithms, and Applications. *Prentice Hall.New Jersey,* 2008.

[14] **D. Rowell.** Data resampling: interpolation (upsampling) and decimation (down-sampling). *Massachusetts institute of technology department of mechanical engineering. 2.161. Signalprocessing - continuous and discrete.* 2008, 1-11, http://ocw.mit.edu/terms.

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