# On the Max-Sum Equivalence in Presence of Negative Dependence and Heavy Tails*

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crossref http://dx.doi.org/10.5755/j01.itc.44.2.7312

**Abstract**. In this paper, following [1], the equivalence of the tail probabilities for the maximum and the sum with heavy-tailed summands under the negative dependence structure is investigated. Applications to some risk models with financial and insurance risks are provided. The Monte-Carlo simulation study illustrates the results.

**Keywords**: dominatedly varying tails; long-tailed distribution; consistently varying tails; max-sum equivalence; negative dependence.

## 1. Introduction

Assume that $X_1, \ldots, X_n$ are real-valued random variables (r.v.s) with corresponding distributions $F_1, \ldots, F_n$. Denote $S_n := X_1 + \cdots + X_n$ and $S_{(n)} := \max\{S_1, \ldots, S_n\}$. Motivated by the paper of Li and Tang [1] (see also [2]), the aim of this note is to investigate the equivalence among the quantities $\mathrm{P}(S_n > x)$, $\mathrm{P}(\max\{X_1, \ldots, X_n\} > x)$, $\mathrm{P}(S_{(n)} > x)$ and $\sum_{k=1}^{n} \mathrm{P}(X_k > x)$ under some dependence assumption on $X_1, \ldots, X_n$ with nonidentical distributions. Comparing with previous results (see, e.g., [3], [4], [5], [6], [7], [8], [9]), we aim to restrict some conditions to the (heavy-tailed) distribution of $X_{(n)} := \max(X_1, \ldots, X_n)$. The assumption that the r.v.s $X_1, \ldots, X_n$ are nonidentically distributed is important for insurance mathematics, because the result can be applied to some risk models with insurance and financial risks. Namely, set $X_k = \theta_k \xi_k$, where $\xi_k$, $k = 1, \ldots, n$, are real-valued r.v.s, which represent the successive net losses for an insurance company, or can be understood as the total claim amount minus the total premium income within year $k$, and $\theta_k$, $1 \le k \le n$, are nonnegative r.v.s which stand for the discount factor from year $k$ to year $0$. In such a model, the r.v.s $\xi_k$ and $\theta_k$ are called the insurance risk and

* The first author is supported by National Natural Science Foundation of China (No. 71471090, 11001052), the Humanities and Social Sciences Foundation of the Ministry of Education of China (No. 14YJCZH182), China Postdoctoral Science Foundation (No. 2014T70449, 2012M520964), Natural Science Foundation of Jiangsu Province of China (No. BK20131339), Qing Lan Project, PAPD, Project of Construction for Superior Subjects of Statistics of Jiangsu Higher Education Institutions, Project of the Key Lab of Financial Engineering of Jiangsu Province. The second author is supported by a grant (No. MIP-11155) from the Research Council of Lithuania.

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financial risk, respectively (see Section 2), and $\mathrm{P}(S_{(n)} > x) =: \psi(x,n)$ represents the finite-time ruin probability by year $n$ with initial capital $x > 0$. The obtained below asymptotic relations are important not only from the theoretical point of view, but also they can be used in practice as a numerical tool allowing to approximate the ruin probability $\psi(x,n)$ by the tail distribution of the maximal random variable $X_{(n)}$. A small Monte-Carlo simulation study in Section 3 illustrates this approximation.

Throughout the paper all limit relationships hold for $x$ tending to $\infty$. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \le 1$; write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \ge 1$ and write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$.

Recall some important classes of heavy-tailed distributions used in the paper. A d.f. $F = 1 - \overline{F}$ is said to belong to the class dominatedly varying-tailed distributions, denoted by $\mathcal{D}$, if $\limsup \overline{F}(yx)/\overline{F}(x) < \infty$ for any $0 < y < 1$. A slightly smaller class is the consistently varying-tailed distribution class, denoted by $\mathcal{C}$. A d.f. $F$ is said to belong to the class $\mathcal{C}$, if $\lim_{y\nearrow 1} \limsup_{x\to\infty} \overline{F}(yx)/\overline{F}(x) = 1$. A d.f. $F$ belongs to the class of long-tailed distributions, denoted by $\mathcal{L}$, if $\lim \overline{F}(x+y)/\overline{F}(x) = 1$ for any $y \in \mathbb{R}$. The following proper inclusion relationship between the forementioned classes holds:

$$\mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L}.$$

For more details on heavy-tailed distributions, see [10].

Furthermore, for a d.f. $F$, denote its upper Matuszewska index by

$$J_F^+ := -\lim_{y\to\infty} \frac{\log F_*(y)}{\log y}$$

with $\overline{F}_*(y) := \liminf \frac{\overline{F}(yx)}{\overline{F}(x)}$ for $y > 1$.

Another important parameter is $L_F := \lim_{y\searrow 1} \overline{F}_*(y)$. The following assertions are equivalent: (i) $F \in \mathcal{D}$; (ii) $L_F > 0$; (iii) $J_F^+ < \infty$. It also holds that $F \in \mathcal{C}$ if and only if $L_F = 1$. For more details, see [11] (Chapter 2.1).

Next, we recall some concepts of negative dependence, which were introduced by [12] and [13]. R.v.s $Y_1, \dots, Y_n$ are said to be upper negatively dependent (UND) if, for all $x_1, \dots, x_n$,

$$\mathrm{P}(\cap_{k=1}^n \{Y_k > x_k\}) \le \prod_{k=1}^n \mathrm{P}(Y_k > x_k). \qquad (1)$$

Similarly, r.v.s $Y_1, \dots, Y_n$ are said to be lower negatively dependent (LND) if, for all $x_1, \dots, x_n$,

$$\mathrm{P}(\cap_{k=1}^n \{Y_k \le x_k\}) \le \prod_{k=1}^n \mathrm{P}(Y_k \le x_k). \qquad (2)$$

R.v.s $Y_1, \dots, Y_n$ are said to be negatively dependent (ND) if both (1) and (2) hold for all $x_1, \dots, x_n$. $Y_1, \dots, Y_n$ are pairwise negatively dependent (or negatively quadrant dependent, according to [14]), if

$$\mathrm{P}(Y_i > x_i, Y_j > x_j) \le \mathrm{P}(Y_i > x_i)\mathrm{P}(Y_j > x_j) \qquad (3)$$

for all $x_i, x_j \in \mathbb{R}$, $i \ne j$, $i,j \in \{1, \dots, n\}$. It can be shown that (3) is equivalent to

$$\mathrm{P}(Y_i \le x_i, Y_j \le x_j) \le \mathrm{P}(Y_i \le x_i)\mathrm{P}(Y_j \le x_j)$$

for all $x_i, x_j \in \mathbb{R}$, $i \ne j$, $i,j \in \{1, \dots, n\}$. Clearly, for any pairwise ND variables $Y_1, \dots, Y_n$ we have that

$$\mathrm{P}(\max\{Y_1, \dots, Y_n\} > x)$$
$$\ge \textstyle\sum_{k=1}^n \mathrm{P}(Y_k > x)(1 - \sum_{j=1}^n \mathrm{P}(Y_j > x)). \qquad (4)$$

Denote the distribution of $\max\{X_1, \dots, X_n\}$ by $G_n$, $T_n := X_1^+ + \dots + X_n^+$ and $x^+ := \max\{x, 0\}$.

***Proposition 1.*** *Let $X_1, \dots, X_n$ be pairwise ND real-valued r.v.s with corresponding distributions $F_1, \dots, F_n$. If $G_n \in \mathcal{D}$, then*

$$\mathrm{P}(S_{(n)} > x) \le \mathrm{P}(T_n > x) \lesssim \frac{1}{L_{G_n}} \overline{G}_n(x). \qquad (5)$$

*Furthermore, if $G_n \in \mathcal{L} \cap \mathcal{D}$, then*

$$\mathrm{P}(S_{(n)} > x) \le \mathrm{P}(T_n > x) \lesssim \overline{G}_n(x).$$

**Proof.** $G_n \in \mathcal{D}$ implies that $\sum_{k=1}^n \overline{F}_k(x) > 0$ for all $x$ and, using (4),

$$\overline{G}_n(x) \sim \textstyle\sum_{k=1}^n \overline{F}_k(x). \qquad (6)$$

For any $0 < v < 1$ and $x > 0$ write

$$\mathrm{P}(T_n > x) \le \mathrm{P}\left(\bigcup_{k=1}^n \{X_k^+ > (1-v)x\}\right)$$
$$+ \ \mathrm{P}\left(T_n > x, \bigcap_{k=1}^n \{X_k^+ \le (1-v)x\}\right)$$
$$\le \sum_{k=1}^n \overline{F}_k((1-v)x)$$
$$+ \ \mathrm{P}(T_n > x, \bigcup_{i=1}^n \{X_i^+ > x/n\},$$
$$\bigcap_{k=1}^n \{X_k^+ \le (1-v)x\})$$
$$=: I_1(x) + I_2(x).$$

We have that $I_1(x) \lesssim L_{G_n}^{-1} \overline{G}_n(x)$, because, by $G_n \in \mathcal{D}$,

$$\limsup \frac{I_1(x)}{L_{G_n}^{-1} \overline{G}_n(x)}$$
$$\le \limsup \frac{I_1(x)}{\overline{G}_n((1-v)x)} \limsup \frac{\overline{G}_n((1-v)x)}{L_{G_n}^{-1} \overline{G}_n(x)}$$

for any $0 < v < 1$. As for $I_2(x)$, we have

$$I_2(x)$$

$$\leq \sum_{i=1}^{n} \mathrm{P}\left(T_n > x, X_i^+ > \frac{x}{n}, \bigcap_{k=1}^{n} \{X_k^+ \leq (1-v)x\}\right)$$

$$\leq \sum_{i=1}^{n} \mathrm{P}\left(T_n - X_i^+ > vx, X_i^+ > \frac{x}{n}\right)$$

$$\leq \sum_{i=1}^{n} \mathrm{P}\left(\bigcup_{j\neq i} \left\{X_j^+ > \frac{vx}{n-1}\right\}, X_i^+ > \frac{x}{n}\right)$$

$$\leq \sum_{i=1}^{n} \sum_{j\neq i} \mathrm{P}\left(X_j^+ > \frac{vx}{n-1}, X_i^+ > \frac{x}{n}\right)$$

$$\leq \sum_{i=1}^{n} \sum_{j\neq i} \overline{F}_j\left(\frac{vx}{n-1}\right)\overline{F}_i\left(\frac{x}{n}\right),$$

where in the last step we used that $X_1^+, \dots, X_n^+$ are pairwise ND. Hence, by (6) and $G_n \in \mathcal{D}$, we obtain

$$I_2(x) \lesssim \overline{G}_n\left(\frac{vx}{n-1}\right)\overline{G}_n\left(\frac{x}{n}\right) = o\left(\overline{G}_n(x)\right).$$

If $G_n \in \mathcal{L} \cap \mathcal{D}$, then substitute $vx$ in the above proof with $\ell(x)$, where $\ell(x)$ is a positive function satisfying $\ell(x) \to \infty$, $\ell(x) = o(x)$, and

$$\overline{G}_n\big(x - \ell(x)\big) \sim \overline{G}_n(x) \qquad (7)$$

by $G_n \in \mathcal{L}$ (see [8], [15]). In this case, the estimate for $I_2(x)$ remains the same, i.e. $I_2(x) = o(\overline{G}_n(x))$, whereas for $I_1(x)$, due to (7), it holds $I_1(x) \lesssim \overline{G}_n(x)$. ■

In the case where the left-tail is asymptotically dominated by the right-tail, the lower bound can be obtained as well.

***Proposition 2.*** *Let $X_1, \dots, X_n$ be pairwise ND r.v.s. (i) If $G_n \in \mathcal{D}$ and $F_i(-x) = o(\overline{F}_i(x))$ for $i = 1, \dots, n$, then*

$$\mathrm{P}(S_{(n)} > x) \;\geq\; \mathrm{P}(S_n > x) \;\gtrsim\; L_{G_n}\overline{G}_n(x). \qquad (8)$$

*(ii) If $G_n \in \mathcal{C}$ and $F_i(-x) = o(\overline{F}_i(x))$ for $i = 1, \dots, n$, then*

$$\mathrm{P}(S_{(n)} > x) \;\geq\; \mathrm{P}(S_n > x) \gtrsim \overline{G}_n(x). \qquad (9)$$

*(iii) If $G_n \in \mathcal{L} \cap \mathcal{D}$ and $F_i(A) = 0$ for some finite $A < 0$, $i = 1, \dots, n$, then relations in (9) hold.*

**Proof.** (i) For any $v > 0$

$$\mathrm{P}(S_n > x)$$

$$\geq \mathrm{P}\left(S_n > x, \bigcup_{k=1}^{n} \{X_k > (1+v)x\}\right)$$

$$\geq \sum_{k=1}^{n} \mathrm{P}(S_n > x, X_k > (1+v)x)$$

$$- \sum_{1\leq i<j\leq n} \mathrm{P}(S_n > x, X_i > (1+v)x, X_j > (1+v)x)$$

$$=: I_3(x) - I_4(x).$$

Here, since $X_1, \dots, X_n$ are pairwise ND, we obtain

$$I_4(x) \leq \sum_{1\leq i<j\leq n} \mathrm{P}(X_i > x, X_j > x)$$

$$\leq \left(\sum_{i=1}^{n} \mathrm{P}(X_i > x)\right)^2 = o\left(\overline{G}_n(x)\right)$$

according to (6).

As for $I_3(x)$, we have

$$I_3(x)$$

$$\geq \sum_{k=1}^{n} \mathrm{P}(S_n - X_k > -vx, X_k > (1+v)x)$$

$$\geq \sum_{k=1}^{n} (\mathrm{P}(S_n - X_k > -vx) + \overline{F}_k((1+v)x) - 1)$$

$$\geq \sum_{k=1}^{n} \overline{F}_k((1+v)x) - \sum_{k=1}^{n} \mathrm{P}(S_n - X_k \leq -vx)$$

$$=: I_{31}(x) - I_{32}(x).$$

Here, $I_{31}(x) \gtrsim L_{G_n}\overline{G}_n(x)$. For term $I_{32}(x)$ we have

$$I_{32}(x) = \sum_{k=1}^{n} \mathrm{P}\left(\sum_{\substack{i=1\\ i\neq k}}^{n} (-X_i) \geq vx\right)$$

$$\leq \sum_{k=1}^{n} \mathrm{P}\left(\bigcup_{\substack{i=1\\ i\neq k}}^{n} \left\{-X_i \geq \frac{v}{n-1}x\right\}\right)$$

$$\leq n\sum_{i=1}^{n} F_i\left(-\frac{v}{n-1}\,x\right)$$

$$= o(1)\sum_{i=1}^{n} \overline{F}_i\left(\frac{v}{n-1}\,x\right)$$

$$\sim o(1)\overline{G}_n\left(\frac{v}{n-1}\,x\right)$$

$$= o(\overline{G}_n(x))$$

by $G_n \in \mathcal{D}$.

(ii) Use $L_{G_n} = 1$.

(iii) Again, replacing $vx$ in the proof of assertion (i) by the function $\ell(x)$ given in (7), we have $I_{31}(x) \gtrsim \overline{G_n}(x)$, $I_4(x) = o(\overline{G_n}(x))$, and

$$I_{32}(x) = \sum_{k=1}^{n} \mathrm{P}\left(\sum_{\substack{i=1\\ i\neq k}}^{n} (-X_i) \geq \ell(x)\right)$$

$$\leq \sum_{k=1}^{n} \mathrm{P}\left(\bigcup_{\substack{i=1\\ i\neq k}}^{n} \left\{-X_i \geq \frac{\ell(x)}{n-1}\right\}\right)$$

$$\leq n\sum_{i=1}^{n} F_i\left(-\frac{\ell(x)}{n-1}\right) = 0$$

for large $x$ by the assumption of proposition. This ends the proof. ■

Using Proposition 1 and Proposition 1 (iii), we obtain:

**Corollary 1.** *Let $X_1, \dots, X_n$ be nonnegative pairwise ND r.v.s. If $G_n \in \mathcal{L} \cap \mathcal{D}$, then*

$$\mathrm{P}(S_{(n)} > x) \;=\; \mathrm{P}(S_n > x) \sim \overline{G}_n(x).$$

*Remark 1.* Note that class $\mathcal{D}$ is closed under max operation, i.e. if $F_k \in \mathcal{D}$ for all $k = 1, \dots, n$, then $G_n \in \mathcal{D}$ (the inverse statement obviously does not hold). Moreover, the constant $L_{G_n}$ appearing in Propositions 1 and 1 can be estimated from below as follows:

$$L_{G_n} \geq \left(\sum_{k=1}^{n} \frac{1}{L_{F_k}}\right)^{-1} > 0, \qquad (10)$$

where $L_{F_k} := \lim_{y\searrow 1} \liminf \frac{\overline{F_k}(xy)}{\overline{F_k}(x)}$. To show this, for any $y > 0$ write

$$\frac{\overline{G}_n(xy)}{\overline{G}_n(x)} = \frac{\mathrm{P}(\bigcup_{k=1}^{n}\{X_k > xy\})}{\mathrm{P}(\bigcup_{k=1}^{n}\{X_k > x\})}$$

$$\leq \sum_{k=1}^{n} \frac{\mathrm{P}(X_k > xy)}{\mathrm{P}(X_k > x)},$$

which implies

$$\frac{1}{L_{G_n}} = \lim_{y\nearrow 1} \limsup \frac{\overline{G}_n(xy)}{\overline{G}_n(x)}$$

$$\leq \sum_{k=1}^{n} \lim_{y\nearrow 1} \limsup \frac{\overline{F_k}(xy)}{\overline{F_k}(x)}$$

$$= \sum_{k=1}^{n} \frac{1}{L_{F_k}} < \infty,$$

or (10). Hence, $L_{G_n} > 0$, which is equivalent to $G_n \in \mathcal{D}$.

*Remark 2.* The statement of Corollary 1 holds if $F_k \in \mathcal{C}$ for $k = 1, \dots, n$ and r.v.s $X_1, \dots, X_n$ are nonnegative pairwise ND. To see that $G_n \in \mathcal{C}$, note that for any $x, y$ it holds

$$\frac{\overline{G_n}(xy)}{\overline{G_n}(x)}$$

$$= \frac{\mathrm{P}(\bigcup_{k=1}^{n}\{X_k > xy\})}{\mathrm{P}(\bigcup_{k=1}^{n}\{X_k > x\})}$$

$$\geq \frac{\sum_{k=1}^{n}\overline{F_k}(xy) - \sum_{1\leq i<j\leq n}\mathrm{P}(X_i > xy, X_j > xy)}{\sum_{k=1}^{n}\overline{F_k}(x)}$$

$$\geq \min_{1\leq k\leq n}\left\{\frac{\overline{F_k}(xy)}{\overline{F_k}(x)}\right\} - \frac{\sum_{1\leq i<j\leq n}\overline{F_i}(xy)\overline{F_j}(xy)}{\sum_{k=1}^{n}\overline{F_k}(x)}$$

by pairwise ND property. Hence,

$$1 \geq \lim_{y\searrow 1} \liminf \frac{\overline{G_n}(xy)}{\overline{G_n}(x)}$$

$$\geq \lim_{y\searrow 1} \lim\mathit{inf}\ \min_{1\leq k\leq n}\left\{\frac{\overline{F_k}(xy)}{\overline{F_k}(x)}\right\}$$

$$-\lim_{y\searrow 1} \limsup \sum_{j=1}^{n} \overline{F_j}(xy)$$

$$\geq \min_{1\leq k\leq n}\left\{\lim_{y\searrow 1} \liminf \frac{\overline{F_k}(xy)}{\overline{F_k}(x)}\right\} \;=\; 1.$$

## 2. The Model with Financial and Insurance Risk

In this section we consider the model with financial and insurance risk, mentioned in Section 1, i.e. we study the question when the conditions of the propositions above are satisfied for the $X_k = \theta_k\xi_k$. Lemma 2 below gives a simple condition for $X_1, \dots, X_n$ to be upper or lower negatively dependent.

**Lemma 1** *Assume that $\xi_1, \dots, \xi_n$ are independent, almost surely positive r.v.s, $\theta_1, \dots, \theta_n$ are UND (LND, pairwise ND) r.v.s, independent of $\{\xi_1, \dots, \xi_n\}$. Then $\theta_1\xi_1, \dots, \theta_n\xi_n$ are UND (LND, pairwise ND, respectively).*

**Proof.** Assume that $\theta_1, \dots, \theta_n$ are UND r.v.s. Then

$$\mathrm{P}(\theta_1\xi_1 > x_1, \dots, \theta_n\xi_n > x_n)$$

$$= \int_{(0,\infty)} \dots \int_{(0,\infty)} \mathrm{P}\left(\theta_1 > \frac{x_1}{y_1}, \dots, \theta_n > \frac{x_n}{y_n}\right)$$

$$\mathrm{d}F_{\xi_1}(y_1) \dots \mathrm{d}F_{\xi_n}(y_n)$$

$$\leq \int_{(0,\infty)} \dots \int_{(0,\infty)} \mathrm{P}\left(\theta_1 > \frac{x_1}{y_1}\right) \dots \mathrm{P}\left(\theta_n > \frac{x_n}{y_n}\right)$$

$$\mathrm{d}F_{\xi_1}(y_1) \dots \mathrm{d}F_{\xi_n}(y_n)$$

$$= \mathrm{P}(\theta_1\xi_1 > x_1) \dots \mathrm{P}(\theta_n\xi_n > x_n).$$

The cases of LND and pairwise ND are analogous. ■

We obtain the following proposition.

**Proposition 3.** *Assume that* $\xi_1, \dots, \xi_n$ *are independent, almost surely positive r.v.s from* $\mathcal{D}$. *Assume also that* $\theta_1, \dots, \theta_n$ *are pairwise ND r.v.s, independent of* $\xi_1, \dots, \xi_n$, *such that* $P(\theta_i \in [a,b]) = 1$ *for all* $i = 1, \dots, n$ *and some* $0 < a \le b < \infty$. *Then relations (5) and (8) hold.*

**Proof.** Note that the conditions of the proposition imply

$$G_n(x) = \mathrm{P}(\max\{\theta_1\xi_1, \dots, \theta_n\xi_n\} \le x) \in \mathcal{D}, \quad (11)$$

since, by Remark 1, $\mathrm{P}(\max\{\xi_1, \dots, \xi_n\} \le x) \in \mathcal{D}$ and hence

$$\limsup \frac{\mathrm{P}(\max\{\theta_1\xi_1, \dots, \theta_n\xi_n\} > xy)}{\mathrm{P}(\max\{\theta_1\xi_1, \dots, \theta_n\xi_n\} > x)}$$

$$\le \limsup \frac{\mathrm{P}(b\max\{\xi_1, \dots, \xi_n\} > xy)}{\mathrm{P}(a\max\{\xi_1, \dots, \xi_n\} > x)}$$

$$= \limsup \frac{\mathrm{P}(\max\{\xi_1, \dots, \xi_n\} > xya/b)}{\mathrm{P}(\max\{\xi_1, \dots, \xi_n\} > x)} < \infty.$$

It remains to apply Lemma 2 and Propositions 1–1. ■

Finally note that, in the case $H_k(x) := \mathrm{P}(\xi_k \le x) \in \mathcal{D}$ and $\mathrm{P}(\theta_k \in [a,b]) = 1$, the constant $L_{F_k}$ appearing in (10) can be estimated by the constants defined through the function $\overline{H_{k*}}(y) = \liminf \frac{\mathrm{P}(\xi_k > xy)}{\mathrm{P}(\xi_k > x)}$, $y \ge 1$. It is easy to see that

$$L_{F_k} \ \ge \ \lim_{y \searrow 1} \overline{H_{k*}}(y) \overline{H_{k*}}\left(\frac{b}{a}\right).$$

## 3. Numerical Simulations

In this section we perform some numerical simulations in order to check the accuracy of the asymptotic relations obtained in Corollary 1. We compare the tail probabilities $\mathrm{P}(S_n > x)$ and $\overline{G}_n(x)$ for several values of $x$, assuming that r.v.s $X_k$ are distributed according to the common Pareto law with parameters $\kappa, \beta > 0$:

$$F(x;\kappa,\beta) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\beta}, \quad x \ge 0, \quad (12)$$

which belongs to the class $\mathcal{C} \subset \mathcal{L} \cap \mathcal{D}$. We assume that $\{(X_{2k-1}, X_{2k}), k \ge 1\}$ are independent replications of $(X_1, X_2)$ with the joint distribution

$$F_{X_1,X_2}(x,y) = \max\{\alpha F(x)F(y) + (1-\alpha)(F(x) + F(y) - 1),\ 0\}, \quad (13)$$

with parameter $\alpha \in (0,1)$ (see eq. (4.2.7) in [16]). Since $\mathrm{P}(X_1 > x, X_2 > y) \le \alpha\overline{F}(x)\overline{F}(y)$ for all $x, y$, $X_1$ and $X_2$ are ND r.v.s. Hence, by construction, $X_1, \dots, X_n$ ($n$ – even) are nonnegative pairwise ND r.v.s. Moreover, according to Remark 1, $G_n \in \mathcal{C}$.

For our simulations we choose parameters $\kappa = 1$, $\beta = 2$ and $\alpha = 0.5$. We set $n = 10$ and $x = 100, 500, 1000, 2000$. The procedure of the computation of $\mathrm{P}(S_n > x)$ and $\overline{G_n}(x)$ in Corollary 1 consists of the following steps:

**Step 1.** Assign a value for the variable $x$ and set $m = k = 0$;

**Step 2.** Generate the dependent r.v.s $X_1, \dots, X_n$ from (12) and (13);

**Step 3.** Calculate the sum value and the maximal value of $X_1, \dots, X_n$: $S_n = \sum_{i=1}^{n} X_i$ and $X_{(n)} = \max\{X_1, \dots, X_n\}$;

**Step 4.** Compare the two values $S_n$ and $X_{(n)}$ with $x$: if $S_n > x$, then $m = m + 1$, and if $X_{(n)} > x$, then $k = k + 1$;

**Step 5.** Repeat step 2 through step 4, $N = 2 \times 10^6$ times;

**Step 6.** Calculate the estimates of the two tail probabilities $\mathrm{P}(S_n > x)$ and $\overline{G}_n(x)$ as, respectively, $m/N$ and $k/N$.

For specific values of $x$, the simulated values of $\mathrm{P}(S_n > x)$ and $\overline{G_n}(x)$ are presented in Table 1 below. It can be found from the table, that, the larger $x$ becomes, the smaller the difference between the simulated values of $\mathrm{P}(S_n > x)$ and $\overline{G}_n(x)$ is. Therefore, the approximate relationship in Corollary 1 is reasonable.

**Table 1.** Comparison between the empirical values of $\mathrm{P}(S_n > x)$ and $\overline{G_n}(x)$

| $x$ | $\mathrm{P}(S_n > x)$ | $\overline{G}_n(x)$ |
|---|---|---|
| 100 | 0.002060 | 0.001524 |
| 500 | 0.000125 | 0.000118 |
| 1000 | 0.000013 | 0.000013 |
| 2000 | 0.000004 | 0.000004 |

## Acknowledgement

We are grateful to the anonymous referee for his/her helpful comments and suggestions. We also thank Jonas Šiaulys for several valuable remarks.

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Received June 2014.