ISSN 1392–124X (print), ISSN 2335–884X (online) INFORMATION TECHNOLOGY AND CONTROL, 2014, T. 43, Nr. 2

ADRC Tuning Employing the LQR Approach for Decoupling Uncertain MIMO Systems

P. Teppa-Garran[1], G. Garcia[2]

[1] *Departamento de Procesos y Sistemas, Universidad Simón Bolívar, Valle de Sartenejas, Municipio Baruta, Estado, Miranda, Caracas, Venezuela*
e-mail: pteppa@usb.ve

[2] *CNRS, LAAS, 7 Avenue du Colonel Roche, F-31400 Toulouse, France and Université de Toulouse, LAAS, INSA, Toulouse, France*
e-mail: garcia@laas.fr

crossref http://dx.doi.org/10.5755/j01.itc.43.2.4059

Abstract. Active Disturbance Rejection Control (ADRC) tuning employing the LQR approach is applied for decoupling uncertain MIMO systems. This is done by considering all the coupling and interference interactions between the channels of the system as disturbances, using an Extended State Observer (ESO) to estimate them in real time and then canceling its effect employing the estimate as part of the control signal. The ADRC tuning is essentially a pole-placement technique and the desired performance is indirectly achieved through the location of the closed-loop poles. However, the final choice of these poles becomes a trial-and-error strategy. In contrast with pole-placement, in the LQR method, the desired performance objectives are directly and globally addressed by minimizing a quadratic function of the state and control input.

Keywords: Active Disturbance Rejection Control (ADRC); Extended State Observer (ESO); Linear Quadratic Regulator (LQR); Multiple-Input-Multiple-Output (MIMO) systems; Decoupling.

1. Introduction

All real world systems comprise multiple interacting variables [1]. For this reason, they are usually divided into subsystems, so variables can be grouped into several sets corresponding to each subsystem. These subsystems have some interaction. One form of interaction is called interference; here some variables of a subsystem influence other subsystems. However, these subsystems do not influence variables of the first one. In this form of interaction there is no return path to the system originating the interference. On the other hand, in the coupling interaction, there exists a path of cross-influence so that there is a hidden feedback loop. Ignoring it can lead to instability.

In many industrial plants, the basic extension of classical PID controller design, implementation and tuning is the decentralized approach, where structural concepts are used to decouple the interaction between variables. The use of standard equipment and the ease of hand-tuning or understanding by non-specialist technicians are the main advantages of this approach. The control effort is decomposed into two stages: first to decouple the different subsystems and then to control them. Decoupling or non-interacting control is a popular approach to dealing with control loop interactions. Here, the objective is to eliminate completely the effects of loop interactions. Decoupling control was initially developed for deterministic linear systems. Typical approaches include design of state feedback to reach decoupling of state equation [6], decoupling in frequency domain through inverse Nyquist array [28], decoupling via relative gain array [5, 7, 31], decoupling using Singular Value Decomposition of the transfer function matrix [21], and designing precompensators that transform the controlled transfer function matrix into a diagonal matrix or diagonal dominance [1, 26, 32], where the precompensator can take the form of: dynamic decoupling, steady state decoupling or decoupling at one particular frequency. All these different approaches separate the controlled multivariable system into several Single-Input-Single-Output (SISO) subsystems through a suitable decoupler that depends on accurate process model before controller design. But in practice, it is difficult to reach decoupling control of complex industrial multivariable processes characterized by strongly interactions and uncertainties.

One of the main issues in control is to deal with uncertainties including internal (parameter and unmodeled dynamics) and external (disturbances). However, most uncertainties are not measurable. Hence, how to estimate uncertainties by using the control input and output of the system is a significant problem. Many approaches such as, disturbance accommodation control (DAC) [17, 18], the unknown input observer (UIO) [3, 15], the disturbance observer (DOB) [4, 27, 29] and the extended state observer (ESO) [8, 12, 13, 14] have been proposed to estimate uncertainties from the input-output data. In DAC, UIO and DOB the external disturbance of a linear time-invariant system is estimated and then rejected. DAC and UIO can be viewed as a special case of DOB [27]. The main difference between ESO and DAC, UIO and DOB is that ESO was conceived to deal with nonlinear systems with mixed uncertainties (i.e. unmodeled dynamics and disturbances).

Active Disturbance Rejection Control (ADRC) [8, 12, 13, 14] is a robust control method that does not require a detailed mathematical description of the system.

It is based on the extension of the system model trough a virtual state variable, representing everything that it is not included in the mathematical model of the plant. An estimate of this state provided by an ESO can be used in the control signal to decouple the real perturbation in the plant. It is this inherent capacity of decoupling of the ADRC method that has been employed in the control of MIMO systems. This is done by considering all the coupling and interference interactions between the channels of the system as disturbances, use an ESO to estimate them in real time and then canceling its effect employing the estimate as part of the control signal. This strategy has been used in the control of particular MIMO problems by decomposing the global system into several SISO subsystems and then designing ADRC for each loop, for example: [16, 19, 22, 30, 33], to cite few of them. There are also some contributions that propose a general ADRC framework to treat the MIMO systems: we can mention in this case: [34] where MIMO systems with time delay are considered by viewing the system with time delay in the input as a high-order system without time delay in the input, [24] where it is employed a dynamic decoupling method in the control of a performance turbofan engine and [36] where a dynamic decoupling control based on SISO- ADRC is used for uncertain square MIMO systems with predetermined input-output pairs.

The tuning procedure in ADRC was originally proposed in a nonlinear form [8, 12, 13, 14], but the large number of gains made tuning an art. The structure was simplified to its linear form [9] and parameterized into a few gains. In its linear form, the tuning is essentially a pole-placement technique and the desired performance is indirectly achieved through the location of the closed- loop poles (controller and ESO). However, the final choice of these poles becomes a trial-and-error strategy that may be difficult for practicing engineers to fully understand and to competently apply to real systems. Moreover, the tuning procedure does not address the problem of control effort and in practice all actuators have maximal movement constraints in the form of saturation limits on amplitude. LQR [2, 20] is a well-known design technique in modern optimal control theory and has been widely used in many applications. In contrast with pole-placement, the desired performance objectives are directly addressed by minimizing a quadratic function of the state and control input. The resulting optimal control law has many nice properties, including that of closed-loop stability. Furthermore, by the choice of the weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ it is possible to control the tradeoff between the requirements of regulating the state and the expenditure of control energy.

In this paper, we propose a LQR solution to develop optimal tuning algorithms for decoupling uncertain MIMO systems that have been formulated into the ADRC framework. The method allows computing the gain matrices of the controller and the ESO directly and by considering the system in a global way avoiding the general standard approach of SISO-ADRC design into the MIMO case.

The article is organized as follows. Section 2 considers a new input disturbance formulation for uncertain MIMO system. In section 3, the input disturbance model is used to represent the uncertain MIMO system into the ADRC framework. Section 4 addresses the LQR formulation of the MIMO-ADRC problem and gives some algorithms for optimal tuning the gains of the controller and the ESO. Some empirical guidelines for choosing the controller and ESO bandwidths are also treated. Section 5 develops the application of the method to a heat exchanger.

Notation: Capital bold typeface letters denote matrices and small bold typeface letters denote vectors. $\dot{a} = da/dt$, $\ddot{a} = d^2a/dt^2$, $a^i = d^ia/dt^i$, $A^T = tranpose\ of\ A$, $\mathbb{R}$ is the set of real numbers, $\hat{a} = estimate\ of\ vector\ a$, $rk(A) = rank\ of\ A$, $det(A) = determinant\ of\ matrix\ A$.

2. Input disturbance model of a MIMO system

We consider a MIMO nonlinear time varying plant described exactly in its operating range by the followings implicit coupled input-output equations

$$S_i\left(t, \boldsymbol{w}(t), \boldsymbol{v}_1(t), \dots, \boldsymbol{v}_p(t), \boldsymbol{p}_1(t), \dots, \boldsymbol{p}_m(t)\right) = 0, \quad (1)$$

where $S_i(\cdot)$ for $i = 1, \dots p$ is a sufficiently smooth function of the external vector disturbance $\boldsymbol{w}(t) = [w_1(t) \quad \dots \quad w_q(t)]^T$, and the vectors

$$\boldsymbol{v}_i(t) = \begin{bmatrix} y_i(t) & \dot{y}_i(t) & \dots & y_i^{n_{y_i}}(t) \end{bmatrix}^T \quad (2)$$

$$\boldsymbol{p_k}(t) = \begin{bmatrix} u_k(t) & \dot{u}_k(t) & \dots & u_k^{n_{u_i}}(t) \end{bmatrix}^T \tag{3}$$

where $\boldsymbol{u}(t) = [u_1(t) \quad \dots \quad u_m(t)]^{\mathrm{T}}$ is the control vector and $\boldsymbol{y}(t) = [y_1(t) \quad \dots \quad y_p(t)]^{\mathrm{T}}$ is the controlled output. Assume that for some integer n_i, such that $0 < n_i \leq \mathrm{n}_{\mathrm{y}_i}$, it is verified $\frac{\partial S_i}{\partial y_i(t)^{(n_i)}} \neq 0$. The implicit function theorem yields then locally

$$y_i(t)^{(n_i)} = S_i^*(t, \boldsymbol{w}(t), \bar{\boldsymbol{v}}_1(t), \dots, \bar{\boldsymbol{v}}_p(t), \boldsymbol{p}_1(t), \dots, \boldsymbol{p}_m(t)), \tag{4}$$

with

$$\bar{\boldsymbol{v}}_i(t) = [y_i(t) \quad \dot{y}_i(t) \quad \dots \quad y_i^{n_i-1} \quad y_i^{n_i+1} \dots \quad y_i^{n_{y_i}}].$$

By setting $S_i^*(\cdot) = \bar{f}_i(\cdot) + \boldsymbol{b}_{\alpha_i}\boldsymbol{u}(t)$ in (2), being $\boldsymbol{b}_{\alpha_i} = [b_{\alpha_{i,1}} \quad \cdots \quad b_{\alpha_{i,m}}]$ a real unknown scaling vector of the system that can be approximated by the vector $\boldsymbol{b}_{\widehat{\alpha}_i} = [b_{\widehat{\alpha}_{i,1}} \quad \cdots \quad b_{\widehat{\alpha}_{i,m}}]$, one has

$$y_i(t)^{(n_i)} = f_i + \boldsymbol{b}_{\widehat{\alpha}_i}\boldsymbol{u}(t), \tag{5}$$

where $f_i = \bar{f}_i(\cdot) + (\boldsymbol{b}_{\alpha_i} - \boldsymbol{b}_{\widehat{\alpha}_i})\boldsymbol{u}(t)$ is the input disturbance that represents any difference between the model and the real system. That is, f_i includes the combined effects of unmodeled dynamics, external disturbances and loop interactions.

3. MIMO-ADRC formulation

We fix $n = n_1 = n_2 = \cdots = n_p$ in (5). That is,

$$\begin{gathered} y_1(t)^{(n)} = f_1 + \boldsymbol{b}_{\widehat{\alpha}_1}\boldsymbol{u}(t), \\ \vdots \\ y_p(t)^{(n)} = f_p + \boldsymbol{b}_{\widehat{\alpha}_p}\boldsymbol{u}(t). \end{gathered} \tag{6}$$

Equation (6) can be expressed in compact form by defining the input matrix of the system $\boldsymbol{B}_{\widehat{\alpha}} = \begin{bmatrix} \boldsymbol{b}_{\widehat{\alpha}_1} & \cdots & \boldsymbol{b}_{\widehat{\alpha}_p} \end{bmatrix}^T$, the generalized perturbation vector $\boldsymbol{f} = [f_1 \quad \cdots \quad f_p]^T$ and the *n*-th derivative order output vector $\boldsymbol{y}(t)^{(n)} = \begin{bmatrix} y_1(t)^{(n)} & \dots & y_p(t)^{(n)} \end{bmatrix}^T$, that is

$$\boldsymbol{y}(t)^{(n)} = \boldsymbol{f} + \boldsymbol{B}_{\widehat{\alpha}}\boldsymbol{u}(t). \tag{7}$$

We can remove the scaling matrix $\boldsymbol{B}_{\widehat{\alpha}}$ in (7) by doing $\boldsymbol{B}_{\widehat{\alpha}}\boldsymbol{u}(t) = \boldsymbol{u}_{\widehat{\alpha}}(t)$, the plant equation changes now to

$$\boldsymbol{y}(t)^{(n)} = \boldsymbol{f} + \boldsymbol{u}_{\widehat{\alpha}}(t). \tag{8}$$

The real control input is

$$\boldsymbol{u}(t) = \boldsymbol{B}_{\widehat{\alpha}}^T \left(\boldsymbol{B}_{\widehat{\alpha}}\boldsymbol{B}_{\widehat{\alpha}}^T\right)^{-1} \boldsymbol{u}_{\widehat{\alpha}}(t). \tag{9}$$

Here it is supposed that $p \leq m$ and $rk(\boldsymbol{B}_{\widehat{\alpha}}) = p$ so that the right inverse of matrix $\boldsymbol{B}_{\widehat{\alpha}}$ in (9) exists.

Remark 1: *In the case of square plants $p = m$ the right inverse of $\boldsymbol{B}_{\widehat{\alpha}}$ reduces to its inverse. Moreover, under the assumption of predetermined input-output pairs, it can be defined $\boldsymbol{b}_{\widehat{\alpha}_i} = b_{ii}$ and the inverse of $\boldsymbol{B}_{\widehat{\alpha}}$ would be computed as*

$$\boldsymbol{B}_{\widehat{\alpha}}^{-1} = diag(b_{11}^{-1} \quad \dots \quad b_{pp}^{-1}).$$

Let the state vector be

$$\bar{\boldsymbol{x}}(t) = [\boldsymbol{x}_1^T(t), \boldsymbol{x}_2^T(t), \dots, \boldsymbol{x}_n^T(t)]^T = \left[\boldsymbol{y}(t)^T, \dot{\boldsymbol{y}}(t)^T, \dots, \boldsymbol{y}(t)^{(n-1)^T}\right]^T.$$

The state space model of (8) can be written as

$$\begin{cases} \dot{\bar{\boldsymbol{x}}}(t) = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{B}}(\boldsymbol{f} + \boldsymbol{u}_{\widehat{\alpha}}(t)) \\ \boldsymbol{y}(t) = \bar{\boldsymbol{C}}\bar{\boldsymbol{x}}(t) \end{cases}, \tag{10}$$

where

$$\bar{\boldsymbol{A}} = \begin{bmatrix} \mathbb{O}_p & \mathbb{I}_p & \mathbb{O}_p & \dots & \mathbb{O}_p \\ \mathbb{O}_p & \mathbb{O}_p & \mathbb{I}_p & \dots & \mathbb{O}_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \dots & \mathbb{I}_p \\ \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \dots & \mathbb{O}_p \end{bmatrix}_{(np \times np)}$$

$$\bar{\boldsymbol{B}} = \begin{bmatrix} \mathbb{O}_p \\ \mathbb{O}_p \\ \vdots \\ \mathbb{O}_p \\ \mathbb{I}_p \end{bmatrix}_{(np \times p)} \qquad \bar{\boldsymbol{C}} = [\mathbb{I}_p \quad \mathbb{O}_p \quad \cdots \quad \mathbb{O}_p]_{(p \times np)}.$$

Here $\mathbb{O}_p$ and $\mathbb{I}_p$ are zero and identity matrices respectively. Let the vector $\boldsymbol{x}_{n+1} = \boldsymbol{f}$, the state vector is now $\boldsymbol{x} = [\bar{\boldsymbol{x}}^T, \boldsymbol{f}^T]^T \in \mathbb{R}^{(n+1)p}$ and the state space model of the plant becomes

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}_{\widehat{\alpha}}(t) + \boldsymbol{E}\dot{\boldsymbol{f}}(t)) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \end{cases} \tag{11}$$

$$\boldsymbol{A} = \begin{bmatrix} \bar{\boldsymbol{A}} & \bar{\boldsymbol{B}} \\ \mathbb{O}_{p \times np} & \mathbb{O}_p \end{bmatrix}_{p(n+1) \times p(n+1)} \qquad \boldsymbol{B} = \begin{bmatrix} \bar{\boldsymbol{B}} \\ \mathbb{O}_p \end{bmatrix}_{(n+1)p \times p}$$

$$\boldsymbol{C} = \begin{bmatrix} \bar{\boldsymbol{C}} & \mathbb{O}_p \end{bmatrix}_{p \times p(n+1)} \quad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{0}_{np\, xp} \\ \mathbb{I}_p \end{bmatrix}_{p(n+1) \times p}$$

For system (11) the ESO is designed as follows

$$\begin{cases} \dot{\boldsymbol{z}}(t) = \boldsymbol{A}\boldsymbol{z}(t) + \boldsymbol{B}\boldsymbol{u}_{\widehat{\alpha}}(t) + \boldsymbol{L}(\boldsymbol{y}(t) - \widehat{\boldsymbol{y}}(t)) \\ \widehat{\boldsymbol{y}}(t) = \boldsymbol{C}\boldsymbol{z}(t) \end{cases}, \tag{12}$$

where $\boldsymbol{z} = \left[\widehat{\boldsymbol{x}}^T, \widehat{\boldsymbol{f}}^T\right]^T \in \mathbb{R}^{(n+1)p}$ is an estimate of the state vector $\boldsymbol{x}(t)$ and the generalized perturbation $\boldsymbol{f}$ and $\boldsymbol{L} = [\boldsymbol{L}_1, \boldsymbol{L}_2, \dots, \boldsymbol{L}_{n+1}]^T$ is the observer gain matrix with $\boldsymbol{L}_i \in \mathbb{R}^{p \times p}$ (computation of matrix $\boldsymbol{L}$ will be considered in the next section). As the vector $\boldsymbol{z}_{n+1}(t) \to \boldsymbol{f}(\cdot)$, it is used to actively cancel $\boldsymbol{f}(\cdot)$in (8) by applying

$$\boldsymbol{u}_{\widehat{\alpha}}(t) = [\boldsymbol{u}_0(t) - \boldsymbol{z}_{n+1}(t)]. \tag{13}$$

This control law decouples the system into a set of p parallel n integrator systems

$$\boldsymbol{y}(t)^{(n)} = \boldsymbol{u}_0(t). \quad (14)$$

A Proportional-Derivative (PD) type multivariable controller can now be used, that is

$$\boldsymbol{u}_0(t) = \boldsymbol{K}_1(\boldsymbol{r} - \boldsymbol{z}_1(t)) - \boldsymbol{K}_2\boldsymbol{z}_2(t) - \cdots - \boldsymbol{K}_n\boldsymbol{z}_n(t). \quad (15)$$

where $\boldsymbol{r} = [r_1^* \quad \dots \quad r_p^*]^T$ is the desired set point for the vector output $\boldsymbol{y}(t)$ and $\boldsymbol{K} = [\boldsymbol{K}_1, \boldsymbol{K}_2, \dots, \boldsymbol{K}_n]$ is the controller gain matrix with $\boldsymbol{K}_i \in \mathbb{R}^{p \times p}$ (computation of matrix $\boldsymbol{K}$ will be considered in the next section). Now, we demonstrate the separation principle for MIMO-ADRC closed-loop system.

Theorem 1: *If the estimation error in the ESO is defined as*

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) + \boldsymbol{z}(t), \quad (16)$$

then for the augmented vector $[\dot{\bar{\boldsymbol{x}}}^T \quad \boldsymbol{e}^T]^T$ *the closed-loop MIMO-ADRC is described as*

$$\begin{bmatrix} \dot{\bar{\boldsymbol{x}}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{A}} - \bar{\boldsymbol{B}}\boldsymbol{K} & \bar{\boldsymbol{B}}(\boldsymbol{K} \quad \mathbb{O} \\ \mathbb{O} & \boldsymbol{A} - \boldsymbol{LC} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{e}(t) \end{bmatrix} + \begin{bmatrix} \bar{\boldsymbol{B}}\boldsymbol{K}_1 \\ \mathbb{O} \end{bmatrix} \boldsymbol{r} + \begin{bmatrix} \mathbb{O} \\ \boldsymbol{E} \end{bmatrix} \dot{\boldsymbol{f}}. \quad (17)$$

Proof. *Using* (11) *and* (12), *the estimation error Dynamics of the ESO is given as*

$$\dot{\boldsymbol{e}}(t) = (\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{e}(t) + \boldsymbol{E}\dot{\boldsymbol{f}}(t). \quad (18)$$

By replacing (15) *in* (13), *we can express* $\mathbf{u}_{\hat{\alpha}}(t)$ *as*

$$\boldsymbol{u}_{\hat{\alpha}}(t) = \boldsymbol{K}_1\boldsymbol{r} - [\boldsymbol{K}_{p \times np} \quad \mathbb{I}_p]\boldsymbol{z}(t). \quad (19)$$

Then combining (10), (18) *and* (19) *yields the closed-loop MIMO-ADRC* (17).

From (17), it is straightforward to verify that the eigenvalues of the system matrix of the closed-loop MIMO-ADRC equations are given by the eigenvalues of $(\bar{\boldsymbol{A}} - \bar{\boldsymbol{B}}\boldsymbol{K})$ and $(\boldsymbol{A} - \boldsymbol{LC})$. Since it can be shown that the pair $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}})$ is controllable and the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, the stability of (17) can always be ensured by placing the controller and observer poles appropriately. Moreover, under the assumption of boundedness of $\dot{\boldsymbol{f}}(\cdot)$,the BIBO stability of (17) is assured [35]. This is the case when $\dot{\boldsymbol{f}}(\cdot) = 0$ or its rate of change is small. If the rate of change is not negligible, one can design a generalized extended state observer (GESO) of *p*-th order [25] to estimate the state together with $\boldsymbol{f}, \dot{\boldsymbol{f}}, \ddot{\boldsymbol{f}}, \dots, \boldsymbol{f}^{(p-1)}$. If $\boldsymbol{f}^{(p)}$ is negligible, the closed-loop MIMO-ADRC system (17) will be BIBO stable, enabling to deal the problem of fast varying generalized perturbation within the ADRC framework.

4. LQR formulation of the MIMO-ADRC problem

In order to develop the LQR formulation of the MIMO-ADRC problem we first establish the following result.

Theorem 2: *Let the tracking error be*

$$\boldsymbol{\varepsilon}(t) = \bar{\boldsymbol{r}} - \bar{\boldsymbol{x}}(t), \quad (20)$$

where the vector $\bar{\boldsymbol{r}}$ *is the desired general set point defined as*

$$\bar{\boldsymbol{r}} = [\boldsymbol{r}^T \quad \mathbb{O}^T \quad \dots \quad \mathbb{O}^T]^T, \quad (21)$$

and $\boldsymbol{r} = [r_1^* \quad \dots \quad r_p^*]^T$ *is the desired set point for the vector output* $\boldsymbol{y}(t)$. *We use* $\mathbb{O}$ *to denote a zero vector of dimension* $p \ x \ 1$. *Then the system* (10) *with control law* (15) *takes the form*

$$\dot{\boldsymbol{\varepsilon}}(t) = \bar{\boldsymbol{A}}\boldsymbol{\varepsilon}(t) - \bar{\boldsymbol{B}}\boldsymbol{u}_0(t). \quad (22)$$

Proof. *Replacing* (13) *in* (10) *results in*

$$\dot{\boldsymbol{x}}(t) = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) - \bar{\boldsymbol{B}}\boldsymbol{u}_0(t). \quad (23)$$

The control law in (15) *is written in terms of the real state vector*
$\bar{\boldsymbol{x}}(t) = [\boldsymbol{x}_1^T(t), \boldsymbol{x}_2^T(t), \dots, \boldsymbol{x}_n^T(t)]^T$,
that is

$$\boldsymbol{u}_0(t) = \boldsymbol{K}_1(\boldsymbol{r} - \boldsymbol{x}_1(t)) - \boldsymbol{K}_2\boldsymbol{x}_2(t) - \cdots - \boldsymbol{K}_n\boldsymbol{x}_n(t). \quad (24)$$

Taking the derivative of (20) *and using* (23) *yields*

$$\dot{\boldsymbol{\varepsilon}}(t) = -\bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) - \bar{\boldsymbol{B}}\boldsymbol{u}_0(t) \quad (25)$$

By adding and subtracting the term $\bar{\mathbf{A}}\bar{\mathbf{r}}$ *in* (25) *gives*

$$\dot{\boldsymbol{\varepsilon}}(t) = \bar{\boldsymbol{A}}(\bar{\boldsymbol{r}} - \bar{\boldsymbol{x}}(t)) - \bar{\boldsymbol{B}}\boldsymbol{u}_0(t) - \bar{\boldsymbol{A}}\bar{\boldsymbol{r}}. \quad (26)$$

The Theorem is established because $\bar{\boldsymbol{A}}\bar{\boldsymbol{r}} = 0$ *in* (26).

Remark 2: *It must be remembered that because of the separation principle* (17) *we can always apply* (24) *with the estimate of the state to the real system.*

In order to have the LQR formulation of the ADRC problem given by (22) we define the quadratic cost

$$J = \int_0^\infty [\boldsymbol{\varepsilon}(t)^T\boldsymbol{Q}\boldsymbol{\varepsilon}(t) + \boldsymbol{u}_0^T(t)\boldsymbol{R}\boldsymbol{u}_0(t)]\, dt, \quad (27)$$

where $\boldsymbol{Q}$ and $\boldsymbol{R}$ are, respectively, a positive semidefinite and a positive definite matrices. It is well known that the minimization of (27) gives the state feedback control

$$\boldsymbol{u}_0(t) = -\hat{\boldsymbol{K}}\boldsymbol{\varepsilon}(t) \quad (28)$$

with

$$\hat{\boldsymbol{K}} = \boldsymbol{R}^{-1}(-\bar{\boldsymbol{B}})^T\boldsymbol{P}, \quad (29)$$

where $\boldsymbol{P}$ is the symmetric positive definite solution of the Continuous Algebraic Riccati Equation (CARE) given by

$$\overline{\boldsymbol{A}}^T\boldsymbol{P}+\boldsymbol{P}\overline{\boldsymbol{A}}+\boldsymbol{Q}-\boldsymbol{P}(-\overline{\boldsymbol{B}})\boldsymbol{R}^{-1}(-\overline{\boldsymbol{B}})^T\boldsymbol{P}=0. \quad (30)$$

The matrix gain $\boldsymbol{K}$ in (24) is easily obtained from $\widehat{\boldsymbol{K}}$ in (28) by doing

$$-\widehat{\boldsymbol{K}}\boldsymbol{\varepsilon}(t)=[-\widehat{\boldsymbol{K}}_1 \quad -\widehat{\boldsymbol{K}}_2 \quad \dots \quad -\widehat{\boldsymbol{K}}_n]\big(\overline{\boldsymbol{r}}-\overline{\boldsymbol{x}}(t)\big)= -\widehat{\boldsymbol{K}}_1\big(\boldsymbol{r}-\boldsymbol{x}_1(t)\big)-\big(-\widehat{\boldsymbol{K}}_2\big)\boldsymbol{x}_2(t)-\cdots-\big(-\widehat{\boldsymbol{K}}_n\big)\boldsymbol{x}_n(t).$$

That is, the original control law (24) is obtained via the LQR approach by the choice of the matrix gain

$$\boldsymbol{K}=[\boldsymbol{K}_1,\boldsymbol{K}_2,\dots,\boldsymbol{K}_n]=\big[-\widehat{\boldsymbol{K}}_1,-\widehat{\boldsymbol{K}}_2,\dots,-\widehat{\boldsymbol{K}}_n\big]. \quad (31)$$

Hence $\boldsymbol{K}=-\widehat{\boldsymbol{K}}=\boldsymbol{R}^{-1}\overline{\boldsymbol{B}}^T\boldsymbol{P}$. In Fig. 1, it is shown the MIMO-ADRC configuration where the method LQR is employed for optimal tuning the gains.

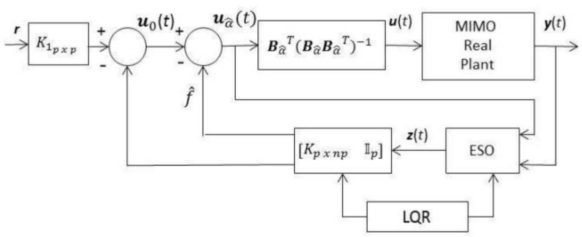

Figure 1. MIMO-ADRC tuning employing LQR

4.1. Selection of matrices Q and R

The matrices $\boldsymbol{Q}\in\mathbb{R}^{np\,x\,np}$ and $\boldsymbol{R}\in\mathbb{R}^{p\,x\,p}$ in (27) are the tuning parameters for computing the matrix $\boldsymbol{K}$ in (31). One typical choice is $\boldsymbol{Q}=\overline{\boldsymbol{C}}^T\overline{\boldsymbol{C}}$ and $\boldsymbol{R}=\lambda\mathbb{I}_p$ with $\lambda>0$, this corresponds to making a trade-off between plant output and input "energies". The quadratic cost (27) takes the form

$$J=\int_0^\infty[\parallel \boldsymbol{y}(t)\parallel^2+\lambda\parallel \boldsymbol{u}_0(t)\parallel^2]\,dt. \quad (32)$$

When λ is small, the convergence $\boldsymbol{y}(t)\to\boldsymbol{r}$ is faster but the control commands $\boldsymbol{u}_0(t)$ are large. When λ is large, the response $\boldsymbol{y}(t)$ is more sluggish and the control commands are smaller.

Remark 3: *With actuator restrictions, we choose λ larger to reduce the control effort at the expense of system performance.*

In LQR, the connection to closed-loop dynamics is indirect; it depends on the choice of matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$. Thus, one usually needs to perform some trial-and-error procedure to obtain satisfactory closed-loop response. For this reason, it is interesting to link LQR to pole placement by requiring that the closed-loop poles of the MIMO-ADRC system (22) with optimal control law (15) and (31) lie in some specific region of the complex plane. A simple example of this is when we require that the closed-loop poles have real part to the left of $s=-\alpha$, for $\alpha\in\mathbb{R}^+$. In LQR theory, this problem is called LQR design with a prescribed degree of stability [2]. In the following result, it is adapted to MIMO –ADRC.

Theorem 3: *Let a MIMO-ADRC system be described by the state equations* (22) *and the LQR criterion*

$$J=\int_0^\infty e^{2w_ct}[\boldsymbol{\varepsilon}(t)^T\boldsymbol{Q}\boldsymbol{\varepsilon}(t)+\boldsymbol{u}_0^T(t)\boldsymbol{R}\boldsymbol{u}_0(t)]\,dt. \quad (33)$$

Then the eigenvalues of the closed-loop matrix $(\overline{\boldsymbol{A}}-\overline{\boldsymbol{B}}\boldsymbol{K})$ lie in $\mathcal{R}eal(s)<-w_c$, where $w_c>0$, and the control signal is $\boldsymbol{u}_0=\boldsymbol{K}\boldsymbol{x}$ with gain matrix $\boldsymbol{K}=\boldsymbol{R}^{-1}\overline{\boldsymbol{B}}^T\boldsymbol{M}$ and $\boldsymbol{M}$ the symmetric positive definite solution of the CARE given by

$$(\overline{\boldsymbol{A}}+w_c\mathbb{I})^T\boldsymbol{M}+\boldsymbol{M}(\overline{\boldsymbol{A}}+w_c\mathbb{I})+\boldsymbol{Q}-\boldsymbol{M}\overline{\boldsymbol{B}}\boldsymbol{R}^{-1}\overline{\boldsymbol{B}}^T\boldsymbol{M}=\mathbb{O}. \quad (34)$$

Proof: *Replacing the coordinate transformation $\boldsymbol{\varepsilon}=e^{w_ct}\boldsymbol{x}$ and $\boldsymbol{v}=e^{w_ct}\mathbf{u}_0$ in* (33) *gives the Standard LQR criterion* (27).

Algorithm 1 shows how to implement Theorem 3 through the Matlab® command lqr.

Algorithm 1

Matlab LQR-MIMO-ADRC design for a prescribed degree of stability $\mathrm{w_c}$

Input:	$\overline{\boldsymbol{A}},\overline{\boldsymbol{B}}$ *from* (10).
	$\boldsymbol{Q},\boldsymbol{R},\mathrm{w_c}$ *from* (33.)
Step 1:	*<<*$\boldsymbol{A}_w=\overline{\boldsymbol{A}}+w_c\,\mathbb{I}_{np}$*; <enter>*
Step 2:	*<<*$\boldsymbol{K}=lgr(\boldsymbol{A}_w,\overline{\boldsymbol{B}},\boldsymbol{Q},\boldsymbol{R})$ *<enter>*
Output:	$\boldsymbol{K}=[\boldsymbol{K}_1 \quad \dots \quad \boldsymbol{K}_n]_{p\,x\,pn}$

Another typical region [11] is when we require the closed-loop poles to be inside a circle with radius ρ and with center at $(-\alpha\,,0)$ with $\alpha>\rho\geq0$. That is, the circle $C(\alpha,\rho)$ is entirely within the left-half plane. This can be achieved by first transforming the Laplace variable s to a new variable ρ, defined as $\sigma=(s+\alpha)/\rho$. This takes the original circle in s-plane to a unit circle in σ-plane. The corresponding, transformed state-space model has the form

$$\sigma\varepsilon(\sigma)=\frac{1}{\rho}(\alpha\mathbb{I}+\overline{\boldsymbol{A}})\varepsilon(\sigma)+\frac{1}{\rho}(-\overline{\boldsymbol{B}})u_0(\sigma)\,. \quad (35)$$

One then treats (35) as the state-space description of a discrete-time system. So, solving the corresponding discrete optimal control problem leads to a feedback matrix such that $(1/\rho)(\alpha\mathbb{I}+\overline{\boldsymbol{A}}+\overline{\boldsymbol{B}}\boldsymbol{K})$ has all its eigenvalues inside the unit disk. This in turn implies that, when the same control law is applied in continuous time, then the closed-loop poles reside in the original circle in s-plane. The above result is summarized in Algorithm 2.

Algorithm 2

Matlab LQR-MIMO-ADRC design such that the closed-loop poles are inside the circle $C(\alpha,\rho)$	
Input:	$\overline{\boldsymbol{A}},\overline{\boldsymbol{B}}$ *from* (10).
	$\boldsymbol{Q},\boldsymbol{R}$ *from* (27.)
Step 1:	<<$\boldsymbol{A}_d=\frac{\overline{\boldsymbol{A}}+\alpha}{\rho}$; <enter>
Step 2:	<< $\boldsymbol{B}_d=\frac{\overline{\boldsymbol{B}}}{\rho}$; <enter>
Step 3:	<<$\boldsymbol{K}=dlgr(\boldsymbol{A}_d,\boldsymbol{B},\boldsymbol{Q},\boldsymbol{R})$ <enter>
Output:	$\boldsymbol{K}=[\boldsymbol{K}_1 \quad \dots \quad \boldsymbol{K}_n]_{p\,\mathrm{x}\,pn}$

Remark 4: *The above ideas can be extended to other cases, in which the desired pole-placement region can be transformed into the stability region, we propose for example* [23].

4.2. LQR-ESO design

ADRC method is based on the separation principle. This allows treating the unknown dynamic and disturbances in a physical process as the generalized disturbance vector $\boldsymbol{f}$ in (8), build an ESO to estimate it in real-time, and then canceling its effect using the estimate as part of the control signal. For computing the ESO gain matrix $\boldsymbol{L}=[\boldsymbol{L}_1,\boldsymbol{L}_2,\dots,\boldsymbol{L}_{n+1}]^T$ with $\boldsymbol{L}_i\in\mathbb{R}^{p\,\mathrm{x}\,p}$ in (12) within a LQR formulation, we propose the LQR design with a prescribed degree of stability [2]. This allows staying in the LQR framework and imposes the practical condition that ESO dynamics must be faster than the controller one. By duality, it is simple to adapt Theorem 1 to design an optimal ESO with a prescribed degree of stability by replacing $A\leftarrow A^T$, $B\leftarrow C^T$ and $K\leftarrow L^T$.

Theorem 4: *We consider an ADRC system where the control law $\boldsymbol{u}_0(t)$ in* (15)*,* (31) *has been computed such that the closed-loop poles are inside the region $\mathcal{Real}(s)<-w_c$ and the quadratic cost of the LQR with a prescribed degree of stability criterion is given by*

$$J=\int_0^\infty e^{2w_0t}(\boldsymbol{z}^T(t)\boldsymbol{Q_0}\boldsymbol{z}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}_0\boldsymbol{u}(t))\,dt. \quad (36)$$

Then the ESO described as (12)*, where w_0 is the required ESO bandwidth chosen as $w_0=\gamma w_c$, has a gain matrix given by*

$$\boldsymbol{L}=\boldsymbol{M}\boldsymbol{C}^T\boldsymbol{R}_0^{-1}\in\mathbb{R}^{np\,\mathrm{x}\,p}, \quad (37)$$

that places the ESO eigenvalues into $\mathcal{Real}(s)<-w_0$ with $\boldsymbol{M}$ the symmetric positive definite solution of the CARE given by

$$(\boldsymbol{A}+w_0\mathbb{I})\boldsymbol{M}+\boldsymbol{M}(\boldsymbol{A}+w_0\mathbb{I})^T+\boldsymbol{Q}_0-\boldsymbol{M}\boldsymbol{C}^T\boldsymbol{R}_0^{-1}\boldsymbol{C}\boldsymbol{M}=\mathbb{O}. \quad (38)$$

Remark 5: *w_0 is chosen γ times the maximal possible closed-loop pole that is defined through w_c in Theorem 1.*

For ease of reference, we summarize the procedure of design in Algorithm 3.

Under the assumption of controllability of the pair $(\overline{\boldsymbol{A}},\overline{\boldsymbol{B}})$ in (10) and by a suitable choice of the feedback matrix $\boldsymbol{K}$ in (15), (31) (Algorithms 1 or 2), the closed-loop poles could be assigned to any desired set of locations. However, if the closed-loop poles are chosen much faster than those of the plant, then the gain $\boldsymbol{K}$ will be large, leading to a large plant input. A similar problem arises in the ESO design (Algorithm 3). If we consider the presence of measurement noise $\boldsymbol{\eta}(t)$ in the controlled output $\boldsymbol{y}(t)$, the estimation error dynamics of the ESO (18) takes the form

$$\dot{\boldsymbol{e}}(t)=(\boldsymbol{A}-\boldsymbol{L}\boldsymbol{C})\boldsymbol{e}(t)+\boldsymbol{E}\dot{\boldsymbol{f}}(t)-\boldsymbol{L}\boldsymbol{\eta}(t). \quad (39)$$

It is evident that a large value for $\boldsymbol{L}$ will enhance the effect of the measurement noise, since this is usually a high-frequency signal. It is then needed a compromise between speed of response and noise immunity. In MIMO-ADRC, for controller design, one places the desired closed-loop poles in the region $\mathcal{Real}(s)<-w_c$ based on typical performance criteria (rise time, settling time, overshoot, etc.). The larger the parametre w_c, the faster the response, the larger the control signal and a system more susceptible to noise. For ESO design, we choose the ESO bandwidth w_0 by fixing the factor γ in $w_0=\gamma w_c$ to be two to six times faster than the controller poles. This ensures the observer error vector vector decays faster than the desired closed-loop dynamic allowing the controller poles to dominate the total response. If there are presence of sensor noise or actuator constraints, the observer poles may be chosen slower than two times w_c. This would yield a system with lower bandwidth, more noise smoothing and less control energy expenditure.

Algorithm 3

Matlab LQR-ESO design with a prescribed degree of stability	
Initialization:	$\boldsymbol{Q_0}=\mathbb{I}_{p(n+1)},\boldsymbol{R_0}=\mathbb{I}_m$
Input:	$\boldsymbol{A},\boldsymbol{C}$ *from* (12)
	$\boldsymbol{Q_0},\boldsymbol{R_0},w_0$ *from* (36.)
Step 1:	<<$\boldsymbol{A}_0=\boldsymbol{A}+w_0\mathbb{I}_{p(n+1)}$ <enter>
Step 2:	<< $\boldsymbol{K}_0=lqr(\boldsymbol{A}_0^T,\boldsymbol{C}^T,\boldsymbol{Q}_0,\boldsymbol{R}_0)$; <enter>
Step 3:	<<$\boldsymbol{L}=\mathrm{transpose}(\boldsymbol{K}_0)$ <enter>
Output:	$\boldsymbol{L}=[\boldsymbol{L}_1,\boldsymbol{L}_2,\dots,\boldsymbol{L}_{n+1}]^T_{p(n+1)\,\mathrm{x}\,p}$

5. Application to a heat exchanger

A heat exchanger is a typical system found in industrial equipment, built for efficient heat transfer from one medium to another. The media may be separated by a solid wall, so that they never mix, or they may be in direct contact. They are widely used in refrigeration, air conditioning, power plants, chemical plants, petrochemical plants, petroleum refineries and natural gas processing. Consider the heat exchanger in Fig. 2 [10]. The lower part is the cold part into which water flows with temperature T_{C_i} and the flow is f_C. The upper part is the hot part with input water temperature T_{H_i} and flow f_H. When the flows meet through separate pipes, the hot water heats the cold water to temperature T_C and it is itself cooled to temperature T_H. By setting up the heat balance in the cold part it is found that the temperatures change according to

$$V_C \dot{T}_C(t) = f_C\left[T_{C_i}(t) - T_C(t)\right] + \phi[T_H(t) - T_C(t)]. \qquad (40)$$

The first term in the right hand side represents the cooling due to the inflow of cold water (normally, $T_{C_i} \le T_C$, so this will give the decrease of temperature). The other term corresponds to the heat transfer from the hot to the cold part of the heat exchanger. It is proportional to the difference in temperature, and the constant of proportionality ϕ, depends on the heat transfer coefficient, the heat capacity of the fluids, etc. Correspondingly, for the hot part one has

$$V_H \dot{T}_H(t) = f_H\left[T_{H_i}(t) - T_H(t)\right] + \phi[T_H(t) - T_C(t)]. \qquad (41)$$

It is now assumed that the flows are constant $f_H = f_C = f$, the outputs are $y_1 = T_C$ and $y_2 = T_H$ and control inputs are selected as $u_1 = T_{C_i}$ and $u_2 = T_{H_i}$. Equations (40) and (41) may be rewritten as

$$\dot{y}_1(t) = \frac{f}{V_C}[u_1(t) - y_1(t)] + \frac{\phi}{V_C}[y_2(t) - y_1(t)], \qquad (42)$$

$$\dot{y}_2(t) = \frac{f}{V_H}[u_2(t) - y_2(t)] + \frac{\phi}{V_H}[y_2(t) - y_1(t)], \qquad (43)$$

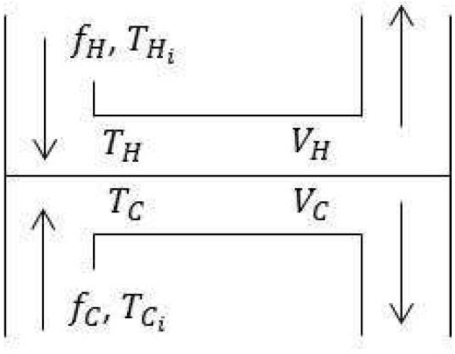

Figure 2. Heat Exchanger

We consider the numerical values $f = 0.01$ m^3/min, $\phi = 0.2\ m^3/min$ and $V_H = V_C = 1\ m^3$. The transfer function matrix is computed as

$$G(s) = \frac{0.01}{(s+0.01)(s+0.41)}\begin{bmatrix} s+0.21 & 0.2 \\ 0.2 & s+0.21 \end{bmatrix} \qquad (44)$$

To measure the degree of interaction between the loops of the system, the concept of relative gain array (RGA) [5] can be used. For (44), the RGA is computed as

$$G(0) \circ [G(0)^{-1}]^T = \begin{bmatrix} 10.76 & -9.76 \\ -9.76 & 10.76 \end{bmatrix}, \qquad (45)$$

where the operator $\circ$ denotes element-wise multiplication. From the values of the RGA in (45), it is evident that the system is strongly coupled.

For MIMO-ADRC design, equations (41) and (42) are written in the form (7) selecting $\boldsymbol{y} = [y_1 \quad y_2]^T = [T_C \quad T_H]^T$, $\boldsymbol{f} = [f_1 \quad f_2]^T$, $\boldsymbol{u} = [u_1 \quad u_2]^T = [T_{C_i} \quad T_{H_i}]^T$ and $\boldsymbol{B}_{\hat{\alpha}} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$. The state space description for $\overline{\boldsymbol{x}}(t) = [\boldsymbol{x}_1^T(t)]^T = [\boldsymbol{y}(t)^T]^T$ is

$$\begin{cases} \dot{\overline{\boldsymbol{x}}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(\boldsymbol{f} + \boldsymbol{u}_{\hat{\alpha}}(t)) \\ \boldsymbol{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\overline{\boldsymbol{x}}(t) \end{cases}. \qquad (46)$$

Comparing (46) with (10) yields

$$\bar{A} = \mathbb{O}_2, \bar{B} = \mathbb{I}_2, \bar{C} = \mathbb{I}_2. \qquad (47)$$

The extended state space description of (46) for $\boldsymbol{x} = [\overline{\boldsymbol{x}}^T, \boldsymbol{f}^T]^T \in \mathbb{R}^4$ is

$$\begin{cases} \dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{u}_{\hat{\alpha}}(t) \\ \qquad + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\dot{\boldsymbol{f}} \\ \boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\boldsymbol{x} \end{cases} \qquad (48)$$

The ESO is obtained through the equations

$$\begin{cases} \dot{\boldsymbol{z}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\boldsymbol{z} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{u}_{\hat{\alpha}} \\ + \begin{bmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ L_{12} & L_{22} & L_{32} & L_{42} \end{bmatrix}^T(\boldsymbol{y} - \hat{\boldsymbol{y}}) \\ \hat{\boldsymbol{y}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\boldsymbol{z}, \end{cases} \qquad (49)$$

where $L = \begin{bmatrix} 12.17 & 0 & 37.5 & 0 \\ 0 & 12.17 & 0 & 37.5 \end{bmatrix}^T$ is the ESO gain matrix which is computed through algorithm 3 for $\boldsymbol{Q}_0 = \mathbb{I}_4, \boldsymbol{R}_0 = \mathbb{I}_2$ and $w_0 = 3$. The control input is

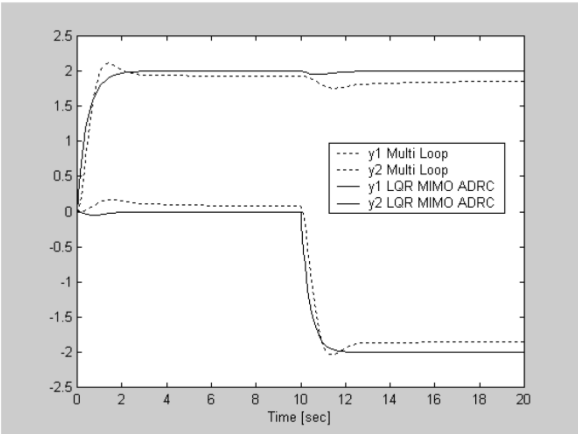

Figure 3. Set-point tracking

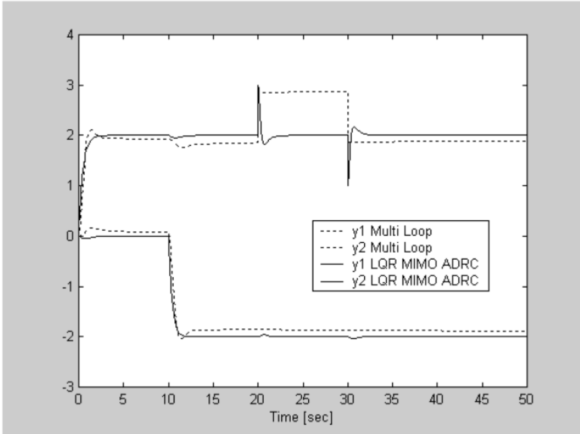

Figure 4. Set-point tracking and disturbance rejection for constant ϕ in (42-43)

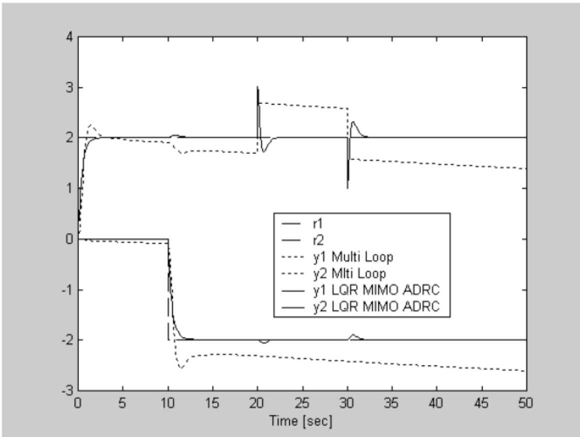

Figure 5. Set-point tracking and disturbance rejection for time-varying ϕ in (42-43)

$$\boldsymbol{u}(t) = \boldsymbol{B}_{\hat{\alpha}}^{-1}\boldsymbol{u}_{\hat{\alpha}}(t) = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \left\{ \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \big(\boldsymbol{r} - \boldsymbol{z}_1(t)\big) - \boldsymbol{z}_2(t) \right\}, \quad (50)$$

where $\boldsymbol{K} = \begin{bmatrix} 2.4142 & 0 \\ 0 & 4.4142 \end{bmatrix}$ is the controller gain matrix and $\boldsymbol{z}_2(t)$ is the estimate of the generalized perturbation $\boldsymbol{f} = [f_1 \quad f_2]^T$. The gain matrix $\boldsymbol{K}$ is computed employing the algorithm 1 by choosing $\boldsymbol{Q} = \mathbb{I}_4, \boldsymbol{R} = \mathbb{I}_2$, and $w_c = 1$. Fig. 3 shows the performance of the proposed method compared with a standard multi-loop design based on steady state decoupling [1, 32] for a set point in channel 1 with amplitude 2 and a set point in channel 2 with amplitude -2 applied since $t = 10\ s$. In Fig. 4, we add an external pulse as a disturbance signal, of amplitude 1 and length 10 seconds since $t = 10\ s$. Finally, in Fig. 5, we repeat the conditions of Fig. 4 but changing the constant ϕ in (41) and (42) into the time varying one $\phi = -0.016(t - 0.2)$.

6. Conclusions

In this paper, a LQR solution has been used to develop optimal tuning algorithms for decoupling uncertain MIMO systems that have been formulated into the ADRC framework. The desired performance objectives are globally and directly addressed by minimizing a quadratic function of the state and control input avoiding the typical trial-and-error strategy of SISO-ADRC for every loop of the MIMO system. The method outperforms the standard multi-loop approach in the case of a heat exchanger.

Acknowledgments

The first author would like to thank the *Laboratoire d'Analyse et d'Architecture des Systèmes* (LAAS – CNRS) for hosting him during the development of this work.

References

[1] **P. Albertos, A. Sala.** Multivariable Control Systems: An Engineering Approach, *Springer*, 2004.

[2] **B. Anderson, J. B. Moore.** Optimal Control: Linear Quadratic Methods, *Prentice-Hall*, 1989.

[3] **G. Basile, G. Marro.** On the observability of linear, time-invariant systems with unknown inputs, *J. of Opt. Th. and Appl.*, 1969, Vol. 2, 410-415.

[4] **R. Bickel, M. Tomizuka.** Passivity-based versus disturbance observer based robot control: equivalence and stability. *ASME J. Dyn. Syst., Meas., and Control*, 1999, Vol. 121, 41-47.

[5] **E. Bristol.** On a new measure of interaction for multivariable process control. *IEEE Trans. Auto. Control*, 1966, Vol. 11, 133-134.

[6] **P. Falb, W. Wolovich.** Decoupling in the design and synthesis of multivariable control systems. *IEEE Trans. Auto. Control*, 1967, Vol. 12, 651-659.

[7] **J. Friedly.** Use of the Bristol array in designing noninteracting control loop: a limitation and extension. *Ind. Eng. Chem. Process Des. Dev*., 1984, Vol. 23, 469-472.

[8] **Z. Gao, Y. Huang, J. Han.** An Alternative Paradigm for Control System. In: *Proc. of IEEE Conference on Decision and Control*, Orlando, USA, 2001, Vol. 5, 4578-4585.

[9] **Z. Gao.** Scaling and Bandwidth Parameterization Based Controller Tuning. In: *Proc. of American Control Conference*, Denver, CO, USA, 2003, 4989-4996.

[10] **T. Glad, L. Ljung.** Control Theory: Multivariable and Nonlinear Methods. *Taylor & Francis*, 2000.

[11] **G. Goodwin, S. Graebe, M. Salgado.** Control System Design, *Prentice Hall*, 2001.

[12] **J. Han.** Auto-Disturbance Rejection Control and its Applications. *Control and Decision*, 1998, Vol. 13, 19-23. (In Chinese).

[13] **J. Han.** Nonlinear Design Methods for Control Systems. In: *Proc. of IFAC World Congress*, Beijing, China, 1999, 521-526.

[14] **J. Han.** From PID to active disturbance rejection control. *IEEE Trans. on Indust. Electronics*, 2009, Vol. 56, 900-906.

[15] **G. Hostetter, J. Meditch.** On the generalization of observers to systems with unmeasurable, unknown inputs. *Automatica*, 1973, Vol. 9, 721-724.

[16] **H. Huang, L. Wu, J. Han, G. Feng, Y. Lin.** A new synthesis method for unit coordinated control system in thermal power plant – ADRC control schema. In: *Proc. of International Conference on Power System Technology*, Singapore, 2004, 133-138.

[17] **C. D. Johnson.** Accommodation of External Disturbances in Linear Regulator and Servomechanism Problems. *IEEE Trans. Automat. Contr*. 1971, Vol. 16, 635-644.

[18] **C. D. Johnson.** Theory of disturbance accommodating controllers. *Control and Dynamic Systems*, 1976, Vol. 12, 387-489.

[19] **F. Khani, A. Yazdizadeh.** Boiler-Turbine unit controller design based on the extended state observer. In: *Proc. of IEEE International Conference on Control and Automation*, New Zealand, 2009, 2066-2071.

[20] **H. Kwakernaak, R. Sivan.** Linear Optimal Control Systems, *Wiley Interscience*, 1972.

[21] **H. Lau, J. Alvarez, K. Jensen.** Synthesis of control structures by singular value analysis: Dynamic measures of sensitive and interaction. *AIChE J.*, 1985, Vol. 31, 427-439.

[22] **B. Liu, Y. Xia, J. Chen.** Decoupling double-loop control system with ADRC. In: *Proc. of International Conference of Innovative Computing Information and Control*, Kaohsiung, Taiwan, 2008, 346-349.

[23] **P. Misra.** LQR design with prescribed damping and degree of stability. In: *Proc. of IEEE International Symposium on Computer-Aided Control System Design*, Michigan, USA, 1996, 68-70.

[24] **R. Miklosovic, Z. Gao.** A dynamic decoupling method for controlling high performance turbofan engines. In: *Proc. of IFAC World Congress*, Czech Republic, 2005, 488-493.

[25] **R. Miklosovic, A. Radke, Z. Gao.** Discrete implementation and generalization of the extended state observer. In: *Proc. of American Control Conference*, Minnesota, USA, 2006, 2209-2214.

[26] **A. Niederlinski.** A heuristic approach to the design of linear multivariable interacting control systems. *Automatica*, 1981, Vol. 7, 691-701.

[27] **J. Profeta, W. Vogt, M. Mickle.** Disturbance estimation and compensation in linear systems. *IEEE Trans. on Aero. and Electr. Syst*., 1990, Vol. 26, 225-231.

[28] **H. Rosenbrock.** Design of multivariable control systems using the inverse Nyquist array. *Proc IEE*, 1969, Vol. 116, No. 11, 1929-1936.

[29] **E. Schrijver, J. Van Dijk.** Disturbance observers for rigid mechanical systems: Equivalence, stability and design. *ASME J. Dyn. Syst., Meas., and Control*, 2002, Vol. 124, 539-548.

[30] **R. Shi, J. Song, W. Wan.** Dynamic modeling and active disturbance rejection controller design for a morphing wing mechanism. In: *Proc. of International Conference on Modeling, Identification and Control*, Wuhan, China, 2012, 1224-1230.

[31] **F. Shinskey.** Process Control Systems, *McGraw-Hill Book Company*, 1979.

[32] **S. Skogestad, I. Postlethwaite.** Multivariable feedback control: Analysis and Design, *Wiley*, 2005.

[33] **L. Tian, D. Li, C. Huang.** Decentralized controller design based on 3-order active-disturbance-rejection-control. In: *Proc. of World Congress on Intelligent Control and Automation*, Beijing, China, 2012, 2746-2751.

[34] **Y. Xia, P. Shi, G. P, Liu, D. Rees, J. Han.** Active disturbance rejection control for uncertain multivariable systems with time-delay. I*ET Control Theory Appl.*, 2007, Vol. 1, 75-81.

[35] **Q. Zheng, L. Gao, Z. Gao.** On stability analysis of active disturbance rejection control for nonlinear time-varying plants with unknown dynamics. In: *Proc. of IEEE Conf. Decision Control*, 2007, 3501-3506.

[36] **Q. Zheng, Z. Chen, Z. Gao.** A practical approach to disturbance decoupling control. *Control Engineering Practice*, 2009, Vol. 17, 1016-1025.

Received April 2013.