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A Novel Approach for Synchronizing of Fractional Order Uncertain Chaotic Systems in the Presence of Unknown Time-Variant Delay and Disturbance

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This paper presents a new method for synchronizing between two fractional order chaotic systems in the simultaneous presence of three categories including uncertainty, external disturbance and time-varying delay. The uncertainties considered in chaotic drive and response systems are on the nonlinear functions, the external disturbances are finite with unknown upper bound, and the delays in the nonlinear functions are 1- variable with time 2- unknown and 3- different from each other in two drive and response systems. A new hybrid method based on fuzzy, adaptive and robust techniques is proposed to achieve synchronization for a specific class of fractional order chaotic systems. The fuzzy method is used to estimate the effects of uncertainties and delayed functions, the adaptive method is employed to obtain the optimal weights of the fuzzy approximator as well as the estimation for upper bound of disturbances, and the robust method is performed to ensure the stability of synchronization and also to cover the errors of both fuzzy and adaptive methods. Simulation in MATLAB environment shows the efficiency of the proposed method in achieving the synchronization goal despite the problems of delay, disturbance and uncertainty.

KEYWORDS. Uncertainty; external disturbance; time-varying delay; synchronization; fractional order chaotic system.

1. Introduction

Most physical systems are nonlinear in nature and exhibit complex dynamic behaviors. One of these nonlinear phenomena is chaotic systems whose behavior is strongly influenced by the initial values and is abundant in chemical reactions, lasers, electronic circuits, as well as in natural phenomena such as the solar system, air and so on. Due to the nature of chaotic systems, they are widely used in various fields such as encryption, secure data transfer, etc. [20, 21, 35, 25, 12].

One of the fields of research related to chaos theory is fractional order systems [9]. Many chaotic systems, such as Lorenz [34], Chen [18], etc. [11] exhibit fractional-order dynamic behavior. Due to the fact that fractional order systems operate more accurately than the integer order type, they are able to describe and model operating systems more accurately and also have a wide range of applications from signal and image processing to automation and robotics control, quantum [4, 15, 24]. Therefore, the chaos theory needs to be further evaluated and developed in this area.

Various researches have been done in relation to fractional order chaotic systems (FOCSs) and numerous methods have been planned to control, stabilize and synchronize them. Given the importance of synchronization [37], which offers tremendous potential for chaotic systems in the areas of secure communication, signal encryption, and fault diagnostics [7, 8, 17], and note that the synchronization of fractional-order systems is more intricate than integer order systems, the synchronization issue of FOCSs is considered in this paper.

To date, several methods have been proposed to achieve synchronization between two chaotic systems, and various control methods have been used for getting the purpose. These methods provide the various types of synchronization, including complete, projective, and lag synchronization [13, 32, 36] using various control methods such as adaptive [31], fuzzy [29], active [5], passive [16], sliding mode [1] and the like. However, there are still fundamental challenges that need to be addressed. The challenge of time delay is a very important issue in chaotic systems like most physical systems. This delay occurs due to the transfer of data, energy or materials and sometimes leads to plant instability. Despite the extensive studies in the field of stability analysis on the issue of synchronization of FOCSs in the presence of time delay, there is still a critical issue, especially when the delay in the system varies with time and is unknown. Another challenge in synchronizing the fractional chaotic systems is the issue of external disturbance. Despite the variety of synchronization methods for integer-order chaotic systems in the presence of disturbances, limited studies have been conducted in the field for fractional-order systems, and at least it can be said that disturbances with unknown upper boundaries are rarely considered. The next challenge that has been considered in this study is the discussion of the existence of uncertainty. The existence of any kind uncertainty strongly affects the synchronization process and it is necessary to find the appropriate measures foe ensuring the stability of the synchronization method.

Considering the mentioned challenges, which include 1- existence of disturbance with unknown upper bound, 2- existence of uncertainty in the system model and 3- existence of unknown time-varying delay, this paper presents a new method for synchronizing a certain class of FOCSs. The upper limit of external disturbance is considered unknown and not only there is uncertainty as the parametric type, but also on the functions. In addition, there are unknown time-varying delay in drive and response systems that make it more difficult to achieve the synchronization goal and they are considered simultaneously with external disturbances and uncertainties in this study. Due to the appropriate efficiency of fuzzy, adaptive and sliding mode methods in different applications [4, 28], an innovative robust control scheme is proposed to synchronize the fractional drive and response systems that combines the above techniques. The fuzzy method is used to estimate the effects of uncertainty and delay in the model, and the adaptive method gives the optimal gains for fuzzy method, in addition, approximates the upper limit of the disturbance. Finally, the sliding mode method makes it possible to achieve robust synchronization and overcome the shortcomings of both fuzzy and adaptive techniques. As many articles [30], stability is guaranteed using the Lyapunov criterion.

Accordingly, this paper has been compiled as follows: In the second part, an introduction is given to fractional relations and fuzzy approximation calculations, and the third part describes the chaotic system class and the innovative synchronization method. In the fourth section, the simulation results are given by applying the proposed method to the fractional chaotic system. Finally, the conclusion of the article is presented in the fifth section.

2. Introduction to Fractional Relations and Calculations and Description of Fuzzy Approximator

In this section, some definitions and relations of fractions and necessary lemmas are given at first and then the fuzzy approximator is described.

2.1. Introduction to Fractional Calculations

Definition 1. [6]. The fractional integrator and derivative operators are as follows:

$$D_t^{\alpha}=\begin{cases}\dfrac{d^{\alpha}}{dt^{\alpha}} & ,\alpha>0\\ 1 & ,\alpha=0,\\ \int_a^t (d\tau)^{-\alpha} & ,\alpha<0\end{cases} \tag{1}$$

where α is a complex number that represents the fractional order.

Definition 2. [2]. Equation (2) shows the fractional order integral of Riemann–Liouville type for function f of order α

$$_{t_0}I_t^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau . \tag{2}$$

The t_0 corresponds to the initial time and the function $\Gamma(\alpha)$ is defined as follows:

$$\Gamma(\alpha)=\int_0^{\infty} e^{-t}t^{\alpha-1}dt, \tag{3}$$

where α indicates the operator of the Gamma function.

Definition 3. [3]. Equation (4) shows the fractional order derivative of Riemann–Liouville type for function f of order α $(n-1<\alpha\le n, n\in N)$

$$_{t_0}D_t^{\alpha} f(t)=\frac{d^{\alpha}f(t)}{dt^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{f(t)}{(t-\tau)^{\alpha-n+1}}d\tau. \tag{4}$$

Definition 4. [26]. Equation (5) shows the Caputo fractional derivative for the continuous function $f(t)$ of order α

$$_{t_0}D_t^{\alpha} f(t)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_{t_0}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-n+1}}, & m-1<\alpha<m,\\ \dfrac{d^m f(t)}{dt^m} & \alpha=m\end{cases} \tag{5}$$

where m is the first integer greater than α.

Remark 1

According to definitions (3) and (4), the Caputo fractional derivative of a constant integer is zero, while the Riemann-Liouville fractional derivative is not zero.

Facts

1- For $\alpha = m$, the function of $D_t^m f(t)$ is the same as $\dfrac{df^m(t)}{dt}$.

2- For $\alpha = 0$, the function of $D_t^{\alpha} f(t)$ is as follows:

$$D_t^0 f(t) = f(t). \tag{6}$$

3- The following linear condition is established for the Caputo fractional order derivative, and accordingly all calculations are performed in terms of the Caputo derivative in this paper:

$$D_t^{\alpha}\left[f(t)+g(t)\right] = D_t^{\alpha} f(t) + D_t^{\alpha}(t). \tag{7}$$

4- The following equation is established for the product of Caputo fractional derivatives of the function $g(t)$:

$$D_t^{\alpha} D_t^{\beta} g(t) = D_t^{\alpha+\beta} g(t). \tag{8}$$

The α and β represent two fraction orders.

The following is a summary of the two lemmas used to design a synchronization method.

Lemma 1. [14]. Consider a fractional non-autonomous system of form (9) where $x = 0$ is the equilibrium point.

$$D^{\alpha} x = f(x,t), \tag{9}$$

where $x = 0$ and $f(x, t)$ satisfies the Lipchitz condition with a factor of $l > 0$. Assuming positive gains for α_1, α_2, α_3 and α, there exists a Lyapunov function that satisfies the following condition

$$\alpha_1 x^{\alpha} \le V(t,x) \le \alpha_2 x \tag{10}$$

$$\dot{V}(t,x) \le -\alpha_3 x. \tag{11}$$

Then, the system is asymptotically stable and this is valid for both Caputo and Riemann-Liouville definitions.

Lemma 2. [22]. The following inequality holds for $x_i \in R, i =, \ldots, n, 0 < q \le 1$:

$$\left(|x_1| + |x_2| + \ldots + |x_n|\right)^q \le |x_1|^q + |x_2|^q + \ldots + |x_n|^q. \tag{12}$$

2.2. Overview of Fuzzy Approximator

A Fuzzy system is specified as a system that provides an outline from the input vector to output vector: $x \to y$, where $x = \left[x_1, \ldots, x_n\right] \in X_1 \times \ldots \times X_n \subseteq R^n, y \in R$.

The fuzzy system has an integrated fuzzification, Gauss membership function, product inference, and principal mean defuzzification, the i_{th} rule of the fuzzy logic system is as follows:

Rule i : *if* x_i *is* F_{i1}, x_n *is* F_{in}, then $y = w_i$, where $i = 1,..., m$, m indicates how many fuzzy logic rules there are, $w_i w_i$ shows the i_{th} fuzzy law, $F_{ij}(j = 1,..., n)$ indicates a fuzzy collection in the world of discourse X_i. Gauss function is as membership function

$$\mu F_{ij}(x_j) = e^{-\left(\frac{x_j - a_{ij}}{b_{ij}}\right)^2}, \tag{13}$$

where a_{ij} and b_{ij} are design components.

To realize the fuzzy system, one should combine the fuzzy rules one after the other, i.e.

$$y(x) = \frac{\sum_{i=1}^{m} w_i \left(\prod_{j=1}^{n} \mu F_{ij}(x_j)\right)}{\sum_{i=1}^{m}\left(\prod_{j=1}^{n} \mu F_{ij}(x_j)\right)} D^{\alpha} x = f(x,t) \tag{14}$$

$$W^T = \left[w_1, \ldots, w_m\right] \tag{15}$$

$$P(x) = \left[p_1(x), p_2(x), \ldots, p_m(x)\right]^T \tag{16}$$

$$p_i(x) = \frac{\prod_{j=1}^{n} \mu F_{ij}(x_j)}{\sum_{i=1}^{m}\left(\prod_{j=1}^{n} \mu F_{ij}(x_j)\right)}. \tag{17}$$

By determining a constant value for membership function, (i.e. a_{ij} and b_{ij} constant), and describing the fuzzy rule w_i as a variable component, we have

$$y(x) = W^T P(x), \tag{18}$$

where $P(x)$ is a fuzzy basis function vector and W is component vector.

There exists a fuzzy system $y^*(x)$ for each real continuous function $y(x)$ in the set $X \subseteq R^n$ and also for each real number $\varepsilon > 0$, that satisfies $\sup |y^*(x) - y(x)| < \varepsilon$, so using the fuzzy system to estimate a continuous function $y(x)$ is offered as follows:

$$f(x) = W^{*T} P(x) + \Delta f(x), \tag{19}$$

where $\Delta f(x)$ contents $\Delta f(x) < \varepsilon$ [10].

3. The Proposed Method for Synchronization of Two Disturbed Uncertain Chaotic Systems of Fractional Order in the Presence of Unknown Time-Varying Delay

In this paper, two n-dimensional disturbed uncertain FOCSs are considered as the drive and response systems. The drive fractional chaotic system is defined as [33]

$$\begin{aligned}
&D^{\alpha} x_1 = f_{11}(x,t) + f_{12}\left(x, x-\tau_1^m(t), t\right) + \Delta f_1(x) + d_1^m(t) \\
&D^{\alpha} x_2 = f_{21}(x,t) + f_{22}\left(x, x-\tau_2^m(t), t\right) + \Delta f_2(x) + d_2^m(t) \\
&\vdots \\
&D^{\alpha} x_n = f_{n1}(x,t) + f_{n2}\left(x, x-\tau_n^m(t), t\right) + \Delta f_n(x) + d_n^m(t),
\end{aligned} \tag{20}$$

where $\alpha \in (0, 1)$ denotes the fractional order of system, $x(t) = [x_1, x_2,..., x_n]^T \in R^n$ is the states of the system $f_{i1}(x, t) \in R$, $i = 1, 2,..., n$ states a specified nonlinear function of t and x. The $f_{i2}(x, x - \tau_i^m(t), t) \in R$, $i = 1, 2,..., n$ is a definite nonlinear function of the states x and time t and the delayed states in which $\tau_i^m(t)$ represents the unknown time variant delay. The $\Delta f_i(x) \in R$, $i = 1, 2,..., n$ determines the unknown parametric uncertainties and $d_i^m(t) \in R$, $i = 1, 2,..., n$ is the unknown bounded disturbances.

There is the following assumption regarding the drive system.

Assumption 1. Suppose that disturbances are in the following form.

$$\left|d_i^m(t)\right| \le D_i^m . i = 1, \ldots, n, \tag{21}$$

where $D_i^m . i = 1, \ldots, n$ are unknown positive constant values. The equations of the response system are also in the following form:

$$\begin{aligned}
&D^{\alpha} y_1 = g_{11}(y,t) + g_{12}\left(y, y-\tau_1^s(t), t\right) + \Delta g_1(y) + d_1^s(t) + u_1(t) \\
&D^{\alpha} y_2 = g_{21}(y,t) + g_{22}\left(y, y-\tau_2^s(t), t\right) + \Delta g_2(y) + d_2^s(t) + u_2(t) \\
&\vdots \\
&D^{\alpha} y_n = g_{n1}(y,t) + g_{n2}\left(y, y-\tau_n^s(t), t\right) + \Delta g_n(y) + d_n^s(t) + u_n(t),
\end{aligned} \tag{22}$$

where $y(t) = [y_1, y_2,..., y_n]^T \in R^n$ signifies the states vector, $g_{i1}(y, t) \in R$, $i = 1, 2,..., n$ determines nonlinear function of t and y that is known. $g_{i2}(y, y - \tau_i^s(t), t) \in R$, $i = 1, 2,..., n$ is a definite nonlinear function of the states y and time t and the delayed states in which $\tau_i^s(t)$ represents the unknown time variant delay. The $\Delta g_i(y) \in R, i = 1, 2,..., n$ indicates the unknown parametric uncertainties, $u_i(t) \in R$, $i = 1, 2,..., n$ is the control signal and $d_i^s(t) \in R$, $i = 1, 2,..., n$ implies the unknown bounded disturbance.

There is the following assumption regarding the response system.

Assumption 2. Suppose that disturbances are in the following form.

$$\left|d_i^s(t)\right| \le D_i^s . i = 1, \ldots, n, \tag{23}$$

where $D_i^s . i = 1, \ldots, n$ are unknown positive constant values. According to the drive and response systems, the synchronization error vector is defined as

$$e_i = y_i - x_i, i = 1,..., n. \tag{24}$$

Using dynamics Equations (20) and (22), the synchronization error becomes as

$$\begin{aligned}
&D^{\alpha} e_1 = g_{11}(y,t) - f_{11}(x,t) + g_{12}\left(y, y-\tau_1^s(t), t\right) - f_{12}\left(x, x-\tau_1^m(t), t\right) \\
&+\Delta g_1(y) - \Delta f_1(x) + d_1^s(t) - d_1^m(t) + u_1(t) \\
&D^{\alpha} e_2 = g_{21}(y,t) - f_{21}(x,t) + g_{22}\left(y, y-\tau_2^s(t), t\right) - f_{22}\left(x, x-\tau_2^m(t), t\right) \\
&+\Delta g_2(y) - \Delta f_2(x) + d_2^s(t) - d_2^m(t) + u_2(t) \\
&\vdots \\
&D^{\alpha} e_n = g_{n1}(y,t) - f_{n1}(x,t) + g_{n2}\left(y, y-\tau_1^s(t), t\right) - f_{n2}\left(x, x-\tau_2^m(t), t\right) \\
&+\Delta g_n(y) - \Delta f_n(x) + d_n^s(t) - d_n^m(t) + u_n(t).
\end{aligned} \tag{25}$$

One can rewrite the error system (25) as follows:

$$\begin{aligned}
&D^{\alpha} e_i = h_{i1}(x,y,t) + h_{i2}\left(x, y, x-\tau_i^m(t), y-\tau_i^s(t), t\right) + \\
&\Delta h_i(x,y) + d_i(t) + u_i(t),
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
&h_{i1}(x,y,t) = g_{i1}(y,t) - f_{i1}(x,t) \\
&h_{i2}\left(x, y, x-\tau_i^m(t), y-\tau_i^s(t), t\right) = \\
&g_{i2}\left(y, y-\tau_i^s(t), t\right) - f_{i2}\left(x, x-\tau_i^m(t), t\right) \\
&\Delta h_i(x,y) = \Delta g_i(t) - \Delta f_i(x) \\
&d_i(t) = d_i^s(t) - d_i^m(t).
\end{aligned} \tag{27}$$

Assumption 3. Error dynamics disturbances are in the following form.

$$\left|d_i(t)\right| \le D_i, i = 1, \ldots, n, \tag{28}$$

where D_i, $i = 1, \ldots, n$ are unknown positive constant values.

Theorem 1. Consider error system (26). This system can be stabilized with the following control signal

$$u_i = -h_{i1}(x, y, t) - k_i e_i - l_i \, sign(e_i) - \hat{W}_i^T P_i(x, y) - \hat{D}_i \tanh\left(\frac{e_i}{\eta_i}\right). \tag{29}$$

As a result, the two systems (20) and (22) are synchronized and the error will be zero between the two response and drive systems.

$$e_i = y_i - x_i = 0. \tag{30}$$

Proof

Consider the following non-negative function as Lyapunov's candidate:

$$V(t) = \frac{1}{2}\sum_{i=1}^{n} D^{\alpha-1} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}\tilde{W}_i^T\tilde{W}_i + \frac{1}{2}\sum_{i=1}^{n}\tilde{D}_i^2, \tag{31}$$

where $\tilde{D}_i = D_i - \hat{D}_i$ and $\hat{D}_i$ is the estimation for disturbances upper bound, and $\tilde{W}_i$ is defined as follows:

$$\tilde{W}_i = W_i - \hat{W}_i, \tag{32}$$

where W_i is the weights of fuzzy approximator and $\tilde{W}_i$ indicates the estimation for the weights of fuzzy approximator. Derived from Equation (31) gives

$$\dot{V}(t) = \sum_{i=1}^{n} D^{\alpha} e_i e_i - \sum_{i=1}^{n}\frac{1}{\lambda_i}\tilde{W}_i^T\dot{\hat{W}}_i - \sum_{i=1}^{n}\tilde{D}_i\dot{\hat{D}}_i$$

$$\dot{V}(t) = \sum_{i=1}^{n}\left[\begin{array}{l} e_i(h_{i1}(x, y, t) + h_{i2}(x, y, x - \tau_i^m(t), y - \tau_i^s(t), t) \\ +\Delta h_i(x, y) + d_i(t) + u_i(t) - \frac{1}{\lambda_i}\tilde{W}_i^T\dot{\hat{W}}_i - \tilde{D}_i\dot{\hat{D}}_i \end{array}\right]. \tag{33}$$

By defining the following function:

$$\varphi_i(x, y, t) = h_{i2}(x, y, x - \tau_i^m(t), y - \tau_i^s(t), t) + \Delta h_i(x, y). \tag{34}$$

Equation (33) is rewritten as follows:

$$\dot{V}(t) = \sum_{i=1}^{n}\left[\begin{array}{l} e_i(h_{i1}(x, y, t) + \varphi_i(x, y, t) + d_i(t) + \\ u_i(t)) - \frac{1}{\lambda_i}\tilde{W}_i^T\dot{\hat{W}}_i - \tilde{D}_i\dot{\hat{D}}_i \end{array}\right]. \tag{35}$$

The function $\varphi_i(x, y, t)$ can now be rewritten as follows using a fuzzy approximator.

$$\varphi_i(x, y, t) = W_i^T P_i(x, y) + \varepsilon_i. \tag{36}$$

It is obtained by placing (36) by (35)

$$\dot{V}(t) = \sum_{i=1}^{n} e_i(h_{i1}(x, y, t) + W_i^T P_i(x, y) + \varepsilon_i + d_i(t) + u_i(t) - \frac{1}{\lambda_i}\tilde{W}_i^T\dot{\hat{W}}_i - \tilde{D}_i\dot{\hat{D}}_i. \tag{37}$$

By selecting the control signal as follows:

$$u_i(t) = -h_{i1}(x, y, t) - k_i e_i - l_i \, sign(e_i) - \hat{W}_i^T P_i(x, y) - \hat{D}_i \tanh\left(\frac{e_i}{\eta_i}\right), \tag{38}$$

where η_i is the design parameter. k_i and l_i are also controller gains. Obtained by placing (38) by (37)

$$\dot{V}(t) = \sum_{i=1}^{n}\left[\begin{array}{l} -k_i e_i^2 - l_i e_i \, sign(e_i) + e_i\tilde{W}_i^T P_i(x, y) + e_i\varepsilon_i \\ +e_i d_i(t) - \hat{D}_i e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \frac{1}{\lambda_i}\tilde{W}_i^T\dot{\hat{W}}_i - \tilde{D}_i\dot{\hat{D}}_i \end{array}\right]. \tag{39}$$

With mathematical simplification, Equation (39) is rewritten as follows:

$$(t) = \sum_{i=1}^{n}\left[\begin{array}{l} -k_i e_i^2 - l_i e_i \, sign(e_i) + \tilde{W}_i^T\left(e_i P_i(x, y) - -\frac{1}{\lambda_i}\dot{\hat{W}}_i\right) \\ +e_i\varepsilon_i + e_i d_i(t) - \hat{D}_i e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \tilde{D}_i\dot{\hat{D}}_i \end{array}\right]. \tag{40}$$

By selecting the adaptive law for fuzzy system weights as follows:

$$\dot{\hat{W}}_i = \lambda_i e_i P_i(x,y). \tag{41}$$

Equation (40) is rewritten as follows:

$$\dot{V}(t) = \sum_{i=1}^{n}\left[-k_i e_i^2 - l_i e_i\, sign(e_i) + e_i\varepsilon_i + e_i d_i(t) - \hat{D}_i e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \tilde{D}_i \dot{\hat{D}}_i\right]. \tag{42}$$

Let us now simplify the Equation (42)

$$\dot{V}(t) = \sum_{i=1}^{n}\begin{bmatrix}-k_i e_i^2 - l_i e_i\, sign(e_i) + e_i\varepsilon_i + e_i d_i(t) \\ -D_i e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \tilde{D}_i e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \tilde{D}_i \dot{\hat{D}}_i\end{bmatrix} \le \sum_{i=1}^{n}\begin{bmatrix}-k_i e_i^2 - l_i|e_i| + \frac{1}{2}e_i^2 + \frac{1}{2}\varepsilon_i^2 + |e_i|D_i \\ -D_i e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \tilde{D}_i\left(e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \dot{\hat{D}}_i\right)\end{bmatrix}. \tag{43}$$

According to inequality $0 \le |x| - x\tanh\left(\frac{x}{\mu}\right) \le 0.2785\mu$ [23], the above equation becomes

$$\dot{V}(t) \le \sum_{i=1}^{n}\left[-(k_i - 0.5)e_i^2 - l_i|e_i| + \frac{1}{2}\varepsilon_i^2 + 0.2785\eta_i D_i + \tilde{D}_i\left(e_i \tanh\left(\frac{e_i}{\eta_i}\right) - \dot{\hat{D}}_i\right)\right]. \tag{44}$$

By selecting the adaptive law to estimate the upper bound of disturbances as follows:

$$\dot{\hat{D}}_i = e_i \tanh\left(\frac{e_i}{\eta_i}\right). \tag{45}$$

It is obtained

$$\dot{V}(t) \le \sum_{i=1}^{n}\left[-(k_i - 0.5)e_i^2 - l_i|e_i| + \frac{1}{2}\varepsilon_i^2 + 0.2785\eta_i D_i\right]. \tag{46}$$

By defining

$$\sigma_i = \frac{1}{2}\varepsilon_i^2 + 0.2785 l_i D_i, \tag{47}$$

we have

$$\dot{V}(t) \le \sum_{i=1}^{n}\left[-(k_i - 0.5)e_i^2 - l_i|e_i| + \sigma_i\right]. \tag{48}$$

Integrate Equation (48) on the span $\xi \in [0, T]$

$$V(T) - V(0) \le \sum_{i=1}^{n}\left(-\int_0^T\left[(k_i - 0.5)e_i^2 + l_i|e_i|\right]d_\xi + \int_0^T \sigma_i d_\xi\right). \tag{49}$$

Considering $V(T) \geqslant 0$ the following is achieved:

$$\sum_{i=1}^{n}\int_0^T\left[(k_i - 0.5)e_i^2 + l_i|e_i|\right]d_\xi \le V(0) + \sum_{i=1}^{n}\int_0^T \sigma_i d_\xi. \tag{50}$$

According to Equation (50), it is known that the closed system is stable and ultimately bounded by applying control law (38) and adaptive laws (41) and (45). As a result, synchronization of the two systems is guaranteed and the error between the two drive and response systems will be zero.

4. Numerical Example

Simulation in MATLAB environment has been used to show the capability of the proposed method. The recently introduced Fei Yu and Chunhua Wang chaotic system with exponential term as a non-linear part has been used to implement the innovative synchronization method. The system definition is as follows:

$$\begin{aligned}\dot{x} &= a(y - x) \\ \dot{y} &= bx - cxz \\ \dot{z} &= e^{xy} - dz.\end{aligned} \tag{51}$$

Its parameters are a = 10, b = 40, c = 2, d = 2.5, their phase portraits are shown in Figure 1.

The fractional equations of the chaotic system are:

$$\frac{d^\alpha x}{dt^\alpha} = a(y-x)$$
$$\frac{d^\alpha y}{dt^\alpha} = bx - cxz \tag{52}$$
$$\frac{d^\alpha z}{dt^\alpha} = e^{xy} - dz\,.$$

With the values of a = 10, b = 40, c = 2, d = 2.5 and α as the fractional order, the phase portrait representation of this system is shown in Figure 2.

By entering uncertainties, disturbances and delay to system (52), the drive system is in the following general form:

$$\frac{d^\alpha x_1}{dt^\alpha} = a\left(x_2 - x_1\left(t-\tau_1^m(t)\right)\right) + \Delta f_1(x) + d_1^m(t)$$
$$\frac{d^\alpha x_2}{dt^\alpha} = bx_1 - cx_1x_3 + \Delta f_2(x) + d_2^m(t) \tag{53}$$
$$\frac{d^\alpha x_3}{dt^\alpha} = e^{x_1x_2} - dx_3\left(t-\tau_2^m(t)\right) + \Delta f_3(x) + d_3^m(t),$$

where parametric uncertainties and external disturbances for the drive system are defined as follows:

$$\Delta f_1(x) = 0.25\cos(6t)x_1$$
$$\Delta f_2(x) = -0.2\cos(2t)x_2 \tag{54}$$
$$\Delta f_3(x) = 0.15\sin(3t)x_3$$

$$d_1^m(t) = -0.15\sin(t)$$
$$d_2^m(t) = 0.1\sin(3t) \tag{55}$$
$$d_3^m(t) = 0.2\cos(5t).$$

The response system is as follows:

$$\frac{d^\alpha y_1}{dt^\alpha} = a\left(y_2 - y_1\left(t-\tau_1^s(t)\right)\right) + \Delta g_1(y) + d_1^s(t) + u_1(t)$$
$$\frac{d^\alpha y_2}{dt^\alpha} = by_1 - cy_1y_3 + \Delta g_2(y) + d_2^s(t) + u_2(t)$$
$$\frac{d^\alpha y_3}{dt^\alpha} = e^{y_1y_2} - dy_3\left(t-\tau_2^s(t)\right) + \Delta g_3(y) + d_3^s(t) + u_3(t), \tag{56}$$

where parametric uncertainties and external disturbances for the response system are defined as follows:

Figure 1

Phase portraits of chaotic system (51)

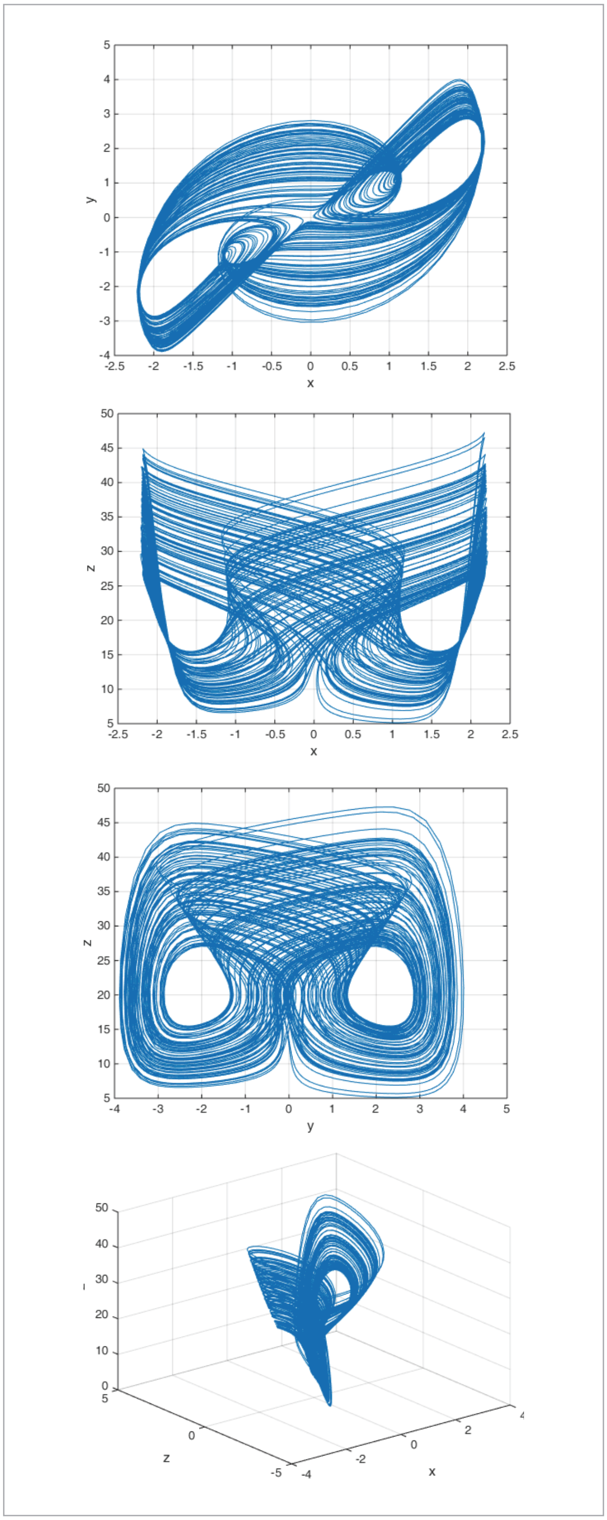
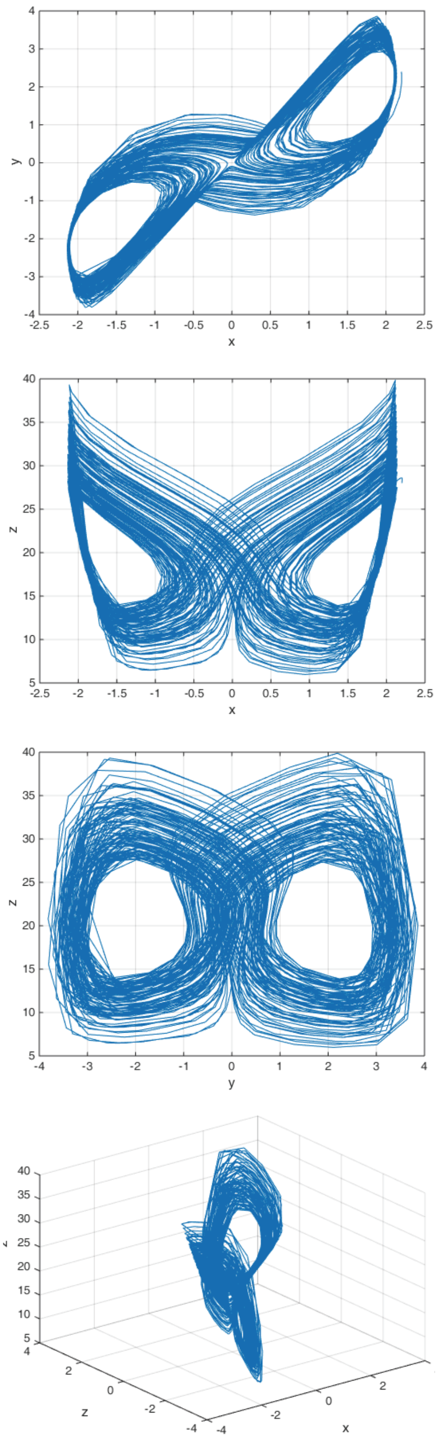


Figure 2
Display of phase portraits for innovative FOCS (52)



$$\begin{aligned}\Delta g_1(y) &= -0.25\sin(4t)\,y_1\\ \Delta g_2(y) &= 0.1\cos(t)\,y_2\\ \Delta g_3(y) &= 0.25\sin(4t)\,y_3\end{aligned} \tag{57}$$

$$\begin{aligned}d_1^s(t) &= 0.1\sin(7t)\\ d_2^s(t) &= 0.15\cos(3t)\\ d_3^s(t) &= -0.15\sin(5t).\end{aligned} \tag{58}$$

$u_1(t)$, $u_2(t)$ and $u_3(t)$ are the control signals obtained from Equation (38) to achieve synchronization between the drive and response systems. The error between the states are defined as follows:

$$\begin{aligned}e_1 &= y_1 - x_1\\ e_2 &= y_2 - x_2\\ e_3 &= y_3 - x_3.\end{aligned} \tag{59}$$

The error dynamics between the two chaotic systems is obtained as follows:

$$\begin{aligned}&\frac{d^\alpha e_1}{dt^\alpha} = a\left(e_2 + y_1\left(t-\tau_1^s(t)\right) - x_1\left(t-\tau_1^m(t)\right)\right)\\ &-0.25\sin(4t)y_1 - 0.25\cos(6t)x_1 + 0.1\sin(7t) + 0.15\sin(t) + u_1\\ &\frac{d^\alpha e_2}{dt^\alpha} = be_1 - c\left(y_1y_3 - x_1x_3\right) + 0.1\cos(t)\,y_2\\ &+0.2\cos(2t)x_2 + 0.15\cos(3t) - 0.1\sin(3t) + u_2\\ &\frac{d^\alpha e_3}{dt^\alpha} = -d\left(y_3\left(t-\tau_2^s(t)\right)x_3\left(t-\tau_2^m(t)\right)\right) + e^{x_2y_2} - e^{x_1y_1}\\ &+0.25\sin(4t)y_3 - 0.15\sin(3t)x_3 - 0.15\sin(5t) - 0.2\cos(5t) + u_3.\end{aligned} \tag{60}$$

Now by setting the initial values as $x_1(0) = 3$, $y_1(0) = 1$, $z_1(0) = 4$ for the drive system and $x_2(0) = 6.2$, $y_2(0) = -1.4$, $z_2(0) = 2$ for the response system and selecting the controller parameters as

$$\begin{aligned}&k_1 = 500, k_2 = 500, k_3 = 500\\ &l_1 = 500, l_2 = 500, l_3 = 500\\ &\eta_1 = 0.01, \eta_2 = 0.01, \eta_3 = 0.01\\ &\lambda_1 = 0.01, \lambda_2 = 0.01, \lambda_3 = 0.01.\end{aligned} \tag{61}$$

The next step is selecting nine membership functions for fuzzy approximator to estimate nonlinear functions as follows:

$$P_i(x) = e^{-(x-0.1i)^2}, \quad i = 1, 2, \ldots, 9. \tag{62}$$

The results of the implementation of the proposed method on the chaotic fractional system are shown in Figures 3-10.

According to the main function of the proposed synchronization method, which covers the variable time delay in different ways in both response and drive systems simultaneously with disturbance and uncertainty, profiles for these delays are shown in Figures 3-6. To examine the ability of the proposed method in more depth, these four different profiles are considered for variable delays with time $\tau_1^m(t)$, $\tau_2^m(t)$, $\tau_1^s(t)$ and $\tau_2^s(t)$.

Figure 3

The time delay ($\tau_1^m(t)$)

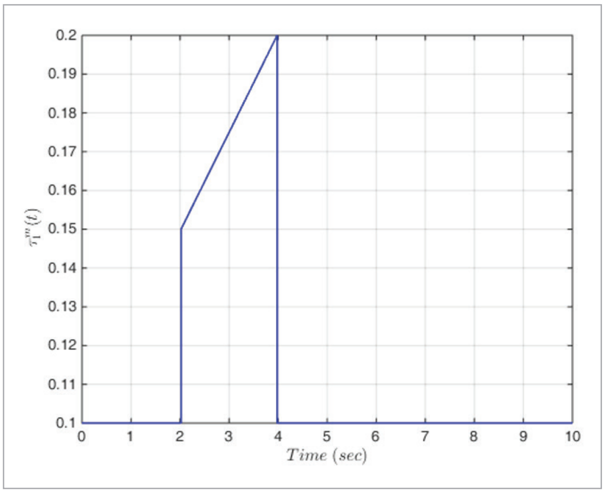


Figure 4

The time delay ($\tau_2^m(t)$)

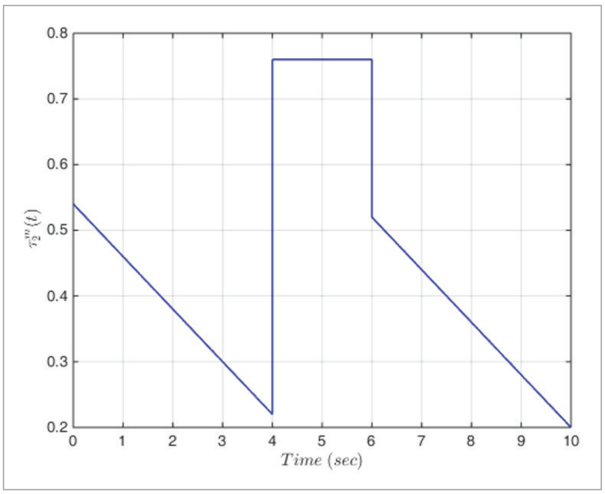


Figure 5

The time delay ($\tau_1^s(t)$)

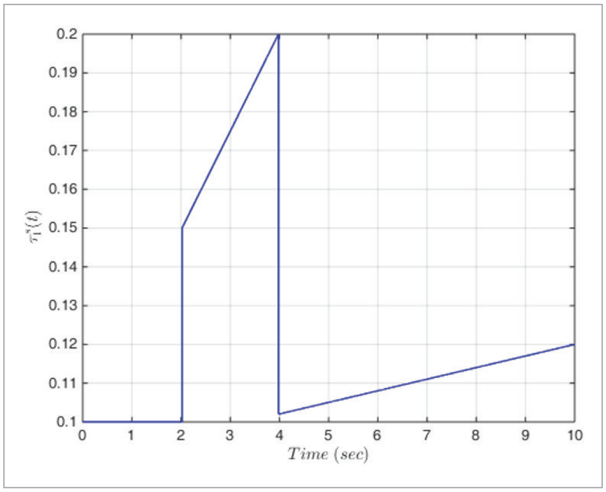


Figure 6

The time delay ($\tau_2^s(t)$)

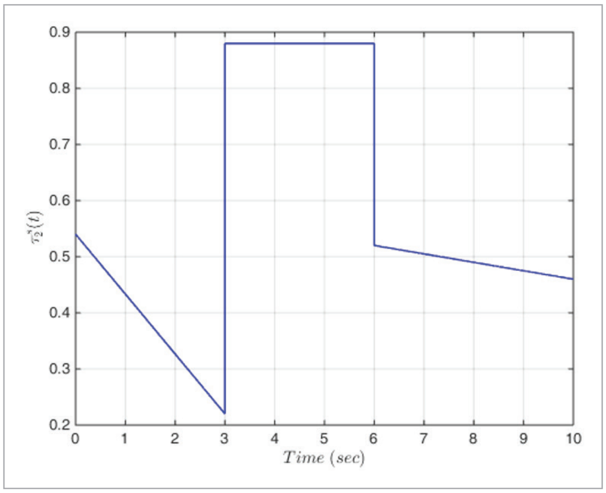


Figure 7 shows the synchronization error resulting from the implementation of the proposed method. As it turns out, after a very limited time of about 0.005s, the two drive and response systems exhibit exactly the same behavior. Also for better evaluation, the error value based on different criteria Integral Square Error (ISE), Integral Time Square Error (ITSE), Integral Absolute Error (IAE) and Integral Time Absolute Error (ITAE) is given in Table 1. By showing the behavior of each state in both drive and response systems, this synchronization is shown in more detail in Figures 8-10. Figure 8 shows the behavior of x_1, y_1, Figure 9 shows the behavior of x_2, y_2 and Figure 10

Figure 7

The display of synchronization errors (e_x, e_y, e_z) between drive and response systems

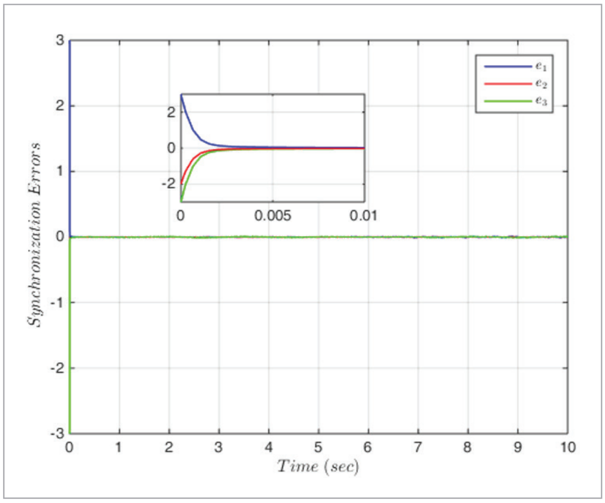


Table 1

The error value based on different criteria

Error	ISE	ITSE	IAE	ITAE
e_1	0.0031	7.62e-4	0.0257	0.1505
e_2	0.0012	5.32e-5	0.0127	0.0408
e_3	0.0032	0.001	0.0395	0.1933

Figure 8

The time-domain display of signal (x_1, y_1) for drive and response systems

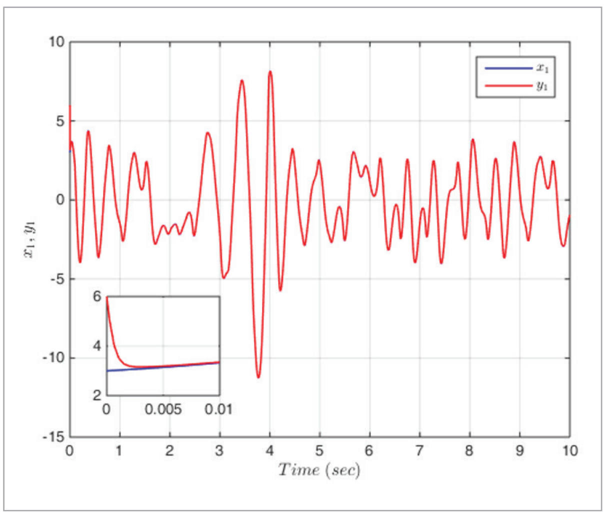


Figure 9

The time-domain display of signal (x_2, y_2) for drive and response systems

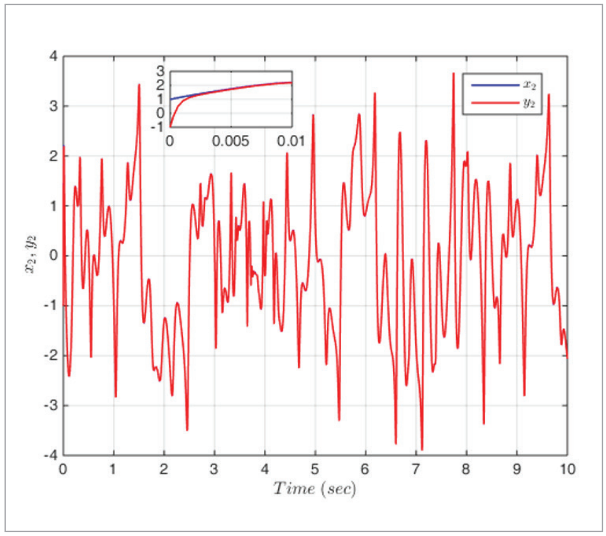
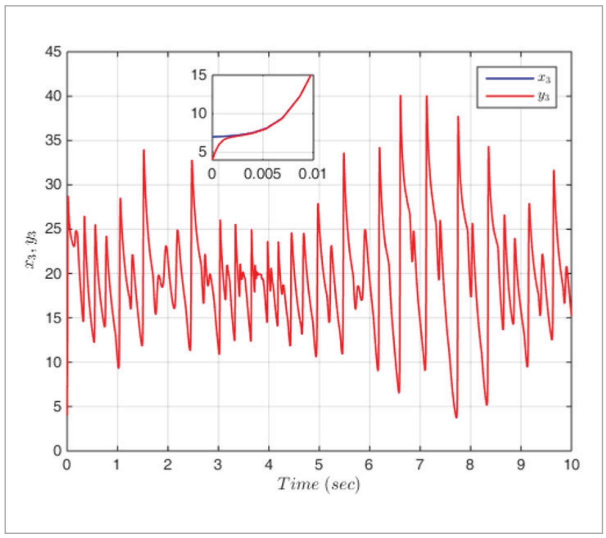


Figure 10

The time-domain display of signal (x_3, y_3) for drive and response systems



shows the behavior of x_3, y_3 in both drive and response systems. The tracking quality of each drive states by the response states is quite evident in these figures. In general, the simulation shows that the proposed method is well able to synchronize two FOCSs in the presence of external disturbances, uncertainties and unknown time-varying delays.

5. Conclusion

In this paper, a new method is presented for synchronizing chaotic systems of fractional order. Three issues affecting the synchronization of the chaotic system including external disturbances, uncertainties and time delays were considered simultaneously. External disturbances were limited but with unknown boundaries, uncertainties could be non-parametric and in addition delay was considered as variable over time, which of course could have unknown boundaries and different forms on drive and response systems. To achieve synchronization, a hybrid control method proposed which included the fuzzy, adaptive and sliding mode techniques. Finally, the simulation in MATLAB environment showed the ability of this controller to achieve the goal of synchronizing two fractional order chaos systems in the shortest time. Optimizing the control signal and considering the constraints on it can be a very good way to complete and develop this study.

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