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Fractals Analysis and Control for a Kind of Three-Species Ecosystem with Symmetrical Coupled Predatory Behavior

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The Lotka-Volterra model plays an important role in the research area of population biology. This work presents the analysis of dynamical behaviours of a kind of three-species Gause-Lotka-Volterra (*GLV* for short) system from the viewpoint of fractals. First, the definition of Julia set which describes the initial distribution rule of the three species' densities is introduced. Second, the gradient control method which contains both giant parameter and state feedback is applied to realize the boundary control of the initial fractals area of three coexisting species. Third, we consider the upper bound of the controlled Julia set from a kind of weakly-coupled *GLV* system, i.e. NPZsystem, by analysing the growth pattern of the initial species. Finally, the nonlinear coupling terms are designed to realize the synchronization of two Julia sets, with the result that the dynamic behaviors of the controlled system can be guided to an ideal one. Numerical examples are included to verify the conclusions of the theoretical investigations.

KEYWORDS: Fractals, Ecosystem, Control, Julia set.

1. Introduction

In the mid-1920s, Lotka and Volterra [15, 28] proposed a pair of differential equations describing the competition between two species, which are now regarded as the theoretical foundation of modern ecology theory. Since proposed, the variants of Lotka-Volterra system and their applications can be founded in diverse areas of biology and even economics [9, 10, 12, 17, 18, 24, 26, 37]. The system were involved in several types from the early ordinary differential equations [10] to the later partial differential equations [9, 12, 31, 32]. In recent years,time-delay [17, 18] was considered in the research of ecosystem aiming to describe the populations growth more effectively. As a kind of nonlinear equations, the research contents of Lotka-Volterra system involved in some classical nonlinear phenomena such as stability of fixed points [1, 3, 5], chaos [1] and bifurcations [4, 14, 38]. Detailed analysis about the chaotic and bifurcation behavior were illustrated via both analytic and numerical methods in [17, 18]. The Hopf bifurcations about a kind of predatorprey model with Michaelis-Menten functional response were systemically investigated in [4, 14, 38]. By applying linear stability analysis method, the authors investigated the conditions of Hopf bifurcation and Turing instability for a kind of diffusive predator-prey model [31] and a spatial plankton model [32]. Among these research processes, the multi-species models have attracted the scholars' attention. One of the multi-species models is *GLV* equation [20, 22] which consists of n differential equations describing the dynamics of n competing populations. *GLV* model is denoted as competing populations. *GLV* model is denoted as

$$\frac{dN_i(t)}{dt} = r_i N_i(t)[1 - \sum_{j=1}^{n} a_{ij} N_j(t)], \tag{1}$$

where $N_i(t)$ is the number of individuals in the ith population at time t, r_i is growth rate of the ith population, and a_{ij} is competition coefficient measuring how the j th species affects the growth rate of the ith species. Robert et al. [20] proposed that the simplest *JLV* equations with even the ideal symmetry predatory behavior could exhibit some complex and nonlinear features.

In view of the discrete feature of population evolution that passes from one generation to the next, the discrete systems in some sense are more accurate than the continuous system in describing the real biological process [6, 23, 39].

In [7], the corresponding discrete version of system (1) was given by making the symmetrical assumptions about the predatory behavior as follows:

$$F: \begin{cases} x_{n+1} = x_n + Rx_n(1 - x_n - \alpha y_n - \beta z_n), \\ y_{n+1} = y_n + Ry_n(1 - \beta x_n - y_n - \alpha z_n), \\ z_{n+1} = z_n + Rz_n(1 - \alpha x_n - \beta y_n - z_n), \end{cases} \tag{2}$$

where x_n, y_n, z_n are the densities of the three species in the nth generation and R represents the linear growth rates. $\alpha_{1,2} = \alpha_{2,3} = \alpha_{3,1} = \beta$ and $\alpha_{2,1} = \alpha_{3,2} = \alpha_{1,3} = \beta$ represent the symmetrical predatory behavior. The stability and bifurcations of fixed points in system were analytically and numerically proposed in [7]. The extension of system (2) to 4D and its applications research can be seen in [2, 16].

Note that for both continuous and discrete version of Lotka-Volterra models, the researches mainly focused on the dynamical behavior of the equilibrium points. In other words, these researches emphatically studied how the system runs when the initial point is given. It is clear that the attractive domain, which shows some fractal characteristics [13, 41, 42], is a set of the initial points of the system which ensures that the trajectories from this set converge to some attractor. Thus the fractals attractive domain is the key to influence the persistence and extinction of the population. Actually, some works have been done by researchers to addressed the problem about the fractals analysis and control for discrete Lotka-Volterra models. In [27], Sun and Zhang gave the definition of Julia set for a kind of discrete Lotka-Volterra system, and realized its control via feedback control and synchronization methods. Some follow-up studies about SIRS model [21] and fractional case [35, 36] were successively given. Nevertheless, there are still some deficiencies about the boundedness analysis of the Julia set of ecosystems. Besides, to our best knowledge, overall previous work on fractals analysis of ecosystems both have not explicitly addressed the issue about the multi-species like system (2).

Motivated by the significative results above, this work focuses on the fractals analysis and control of system (2). The main novelties of this work are summarized as follows: (i). Extend the fractals research on ecosystems to three species case. (ii). A preliminary frame to estimate the boundedness of the Julia sets of multi-species system with chain coupled predatory behavior is given.

The outline of this paper is given as below. In Section 2, the definition and control of the Julia sets of system (2) are proposed. A preliminary study on the boundedness estimation is also given in Section 2. Section 3 illustrates the synchronization process of two Julia sets with different parameters. The summary of this present work and prospect of the future investigations are given in Section 4.

2. Definition and Control of the Julia Sets from Three-Species GLV System

As one of the important branches of fractals, Julia set has attracted the scholars' attentions with respect to the planar case [19, 25, 33] and the spatial case [29, 30, 34]. Julia sets J(f) is defined as the closure of repelling periodic points of a complex function. That is, the trajectories of points in Julia set remain bounded under iterations of f. Particularly, if f has attractive point w, the Julia set can be defined as $J(f) = \partial A(w)$, where A(w) is the attractive domain of the attractive fixed point w. In [27], Sun and Zhang has introduced the

Figure 1

(a).The spacial Julia set $J(F)$ with parameters $R = 1$, $\alpha = 0.4745$, $\beta = 3$. (b). The 2D filled slice K(F) with $z_0 = 0$. (c). The 2D filled slice $K(F)$ with $x_0 = 0.01$. (d). the local enlarge of (c) in which we select point $A = (0.22, 0.19) \in K(F)$ and point $B = (0.23, 0.195) \notin K(F)$

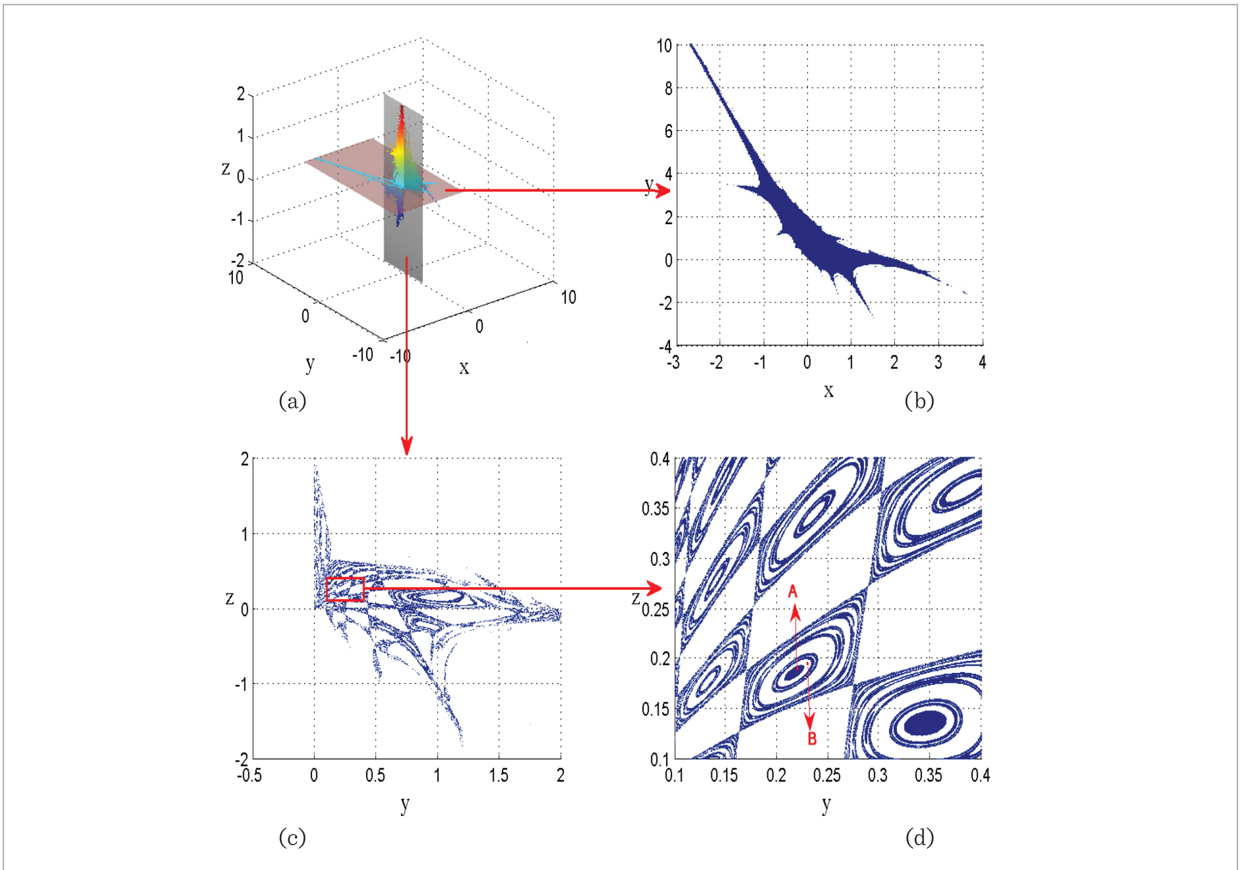


Figure 2

The trajectories of point A of Figure 1 in: (a). 20 steps. (b). 40 steps. (c). 100 steps

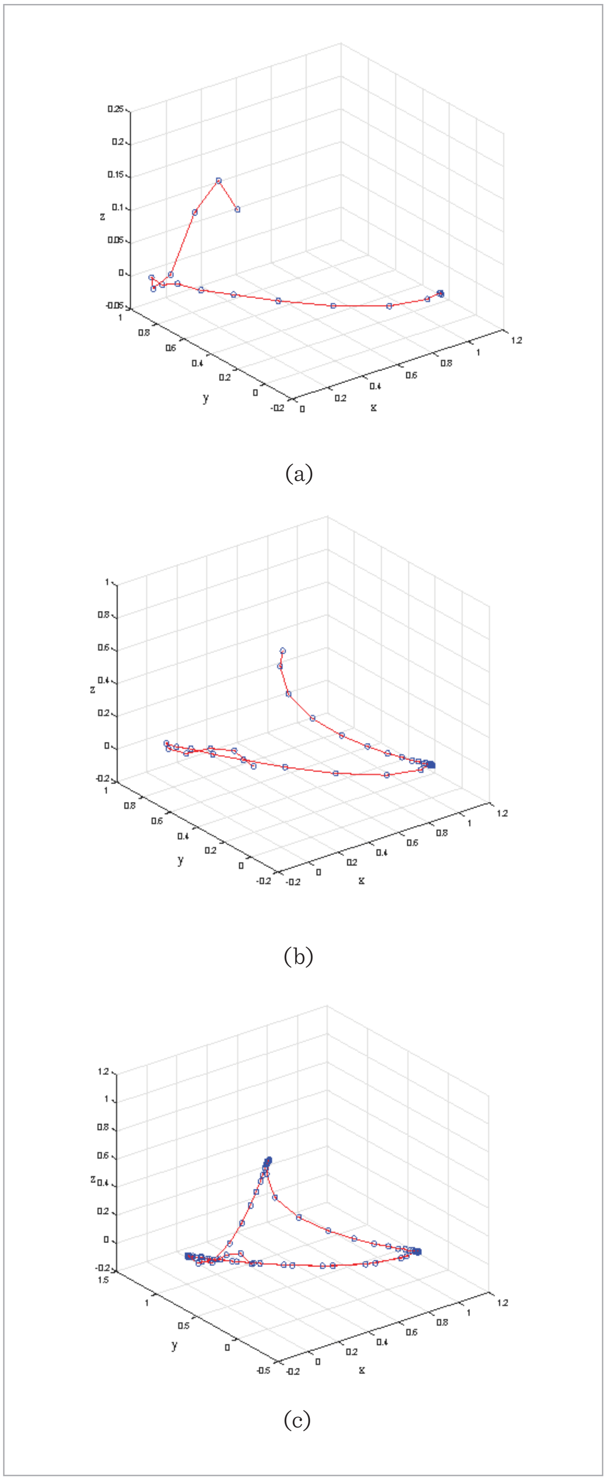
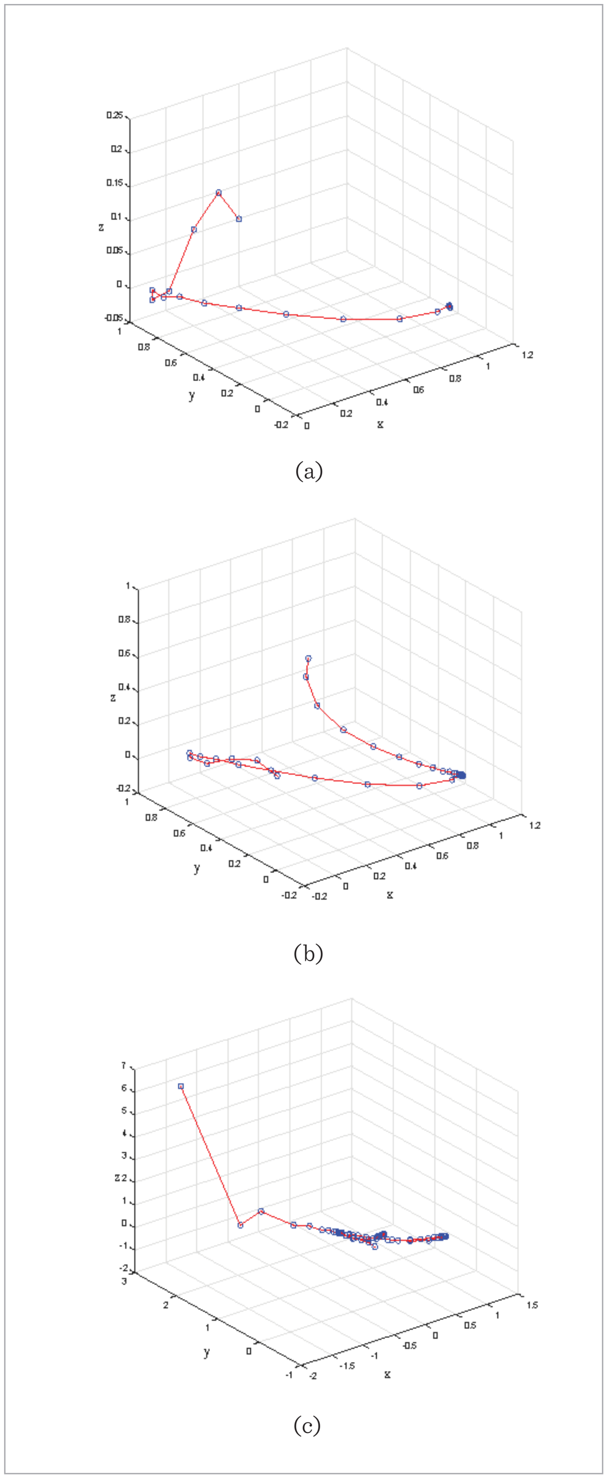


Figure 3

The trajectories of point B of Figure 1 in: (a). 20 steps. (b). 40 steps. (c). 91 steps



definition of Julia set for a 2D discrete Lotka-Volterra system. Then the definition of Julia set J(f) of system (2) is given as follows:

Definition 2.1. Set $\psi_0 = (x_0, y_0, z_0)$ as the initial densities of the three species. The filled Julia set of system (2) is defined as K(F) such that
$K(F) = \{\psi_0 | f^{n\to\infty}(\psi_0)\ remain\ bounded\}$.

While Julia set of system (2) is the boundary of K(F), i. e. J(F) = ∂ K(F).

According to the definition 2.1, the trajectories of initial points and the stability of the fixed points [8] play the key roles for the region of Julia set J(F). Thus the control strategy of Julia set aims to change the trajectories by adding some control items, with result in making the unstable fixed points become local stable and making the points in controlled systems come into its filled Julia set. For system (2), its fixed points (x^*, y^*, z^*) can be summarized into the following eight cases at most:

1 In the case of $(x^*, y^*, z^*) = (0, 0, 0)$, the three species x, y, z are all extinct.

2 When $x_i = 0$ and $y_i, z_i \neq 0$, we can get a fixed point $\left(0, \frac{1+\alpha}{1+\alpha\beta}, \frac{1-\beta}{1+\alpha\beta}\right)$. In this case, species x goes extinct and species y, z lives. Similarly, system (2) has the other two fixed points $\left(\frac{1+\alpha}{1+\alpha\beta}, 0, \frac{1-\beta}{1+\alpha\beta}\right)$ and $\left(\frac{1+\alpha}{1+\alpha\beta}, \frac{1-\beta}{1+\alpha\beta}, 0\right)$.

3 When $x_i = y_i = 0$ and $z_i \neq 0$, we can get a fixed point (0,0,1). In this case, species x,y are extinct and z is independent survival. Similarly, system (2) has the other two fixed points (0,1,0) and (1,0,0).

4 When $x_i, y_i, z_i \neq 0$, we can get the fixed point $\left(\frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}\right)$ that makes three species coexist.

Obviously, we hope that the three groups can coexist sustainedly in the actual biological process. Thus, the control process of the J(F) focuses on the stability of "coexistence point" $\left(\frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}\right)$ In order to keep the fixed points unchanged, we apply the following giant control items [29]:

$$\begin{cases} u_{n1} = -\dfrac{k}{1+k}(f(x_n, y_n, z_n) - x^*), \\ u_{n2} = -\dfrac{k}{1+k}(g(x_n, y_n, z_n) - y^*), \\ u_{n3} = -\dfrac{k}{1+k}(q(x_n, y_n, z_n) - z^*). \end{cases} \tag{3}$$

Because that $x^* = y^* = z^* = \frac{1}{1+\alpha+\beta}$, by adding the control item (3) into system (2), one can get the following controlled system

$$P: \begin{cases} x_{n+1} = \dfrac{1}{1+k}\left(f(x_n, y_n, z_n) + \dfrac{k}{1+k}x^*\right), \\ y_{n+1} = \dfrac{1}{1+k}\left(g(x_n, y_n, z_n) + \dfrac{k}{1+k}x^*\right), \\ z_{n+1} = \dfrac{1}{1+k}\left(q(x_n, y_n, z_n) + \dfrac{k}{1+k}x^*\right). \end{cases} \tag{4}$$

Based on the analysis above, J(F) is the attractive domain of the attractive fixed point. Thus by adding the control items, the Julia set, from which the densities of three species remain stable, can be controlled via affecting the iterative trajectories of initial points in system (4),meanwhile realizing the stability of the fixed point. Consider the Jacobian matrix of system (4) at the fixed point

$(x^*, y^*, z^*) = \left(\frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}\right)$:

$$J = \begin{bmatrix} \frac{1+R}{1+k} - \frac{2R}{1+k}x^* - \frac{\alpha R}{1+k}y^* - \frac{\beta R}{1+k}z^* & -\frac{\alpha R}{1+k}x^* & -\frac{\beta R}{1+k}x^* \\ -\frac{\beta R}{1+k}y^* & \frac{1+R}{1+k} - \frac{2R}{1+k}y^* - \frac{\alpha R}{1+k}z^* - \frac{\beta R}{1+k}x^* & -\frac{\alpha R}{1+k}y^* \\ -\frac{\alpha R}{1+k}z^* & -\frac{\beta R}{1+k}z^* & \frac{1+R}{1+k} - \frac{2R}{1+k}z^* - \frac{\alpha R}{1+k}x^* - \frac{\beta R}{1+k}y^* \end{bmatrix}$$

The characteristic equation of the Jacobi matrix J is denoted as

$\Delta(\lambda) = \lambda^3 + \tilde{P}\lambda^2 + \tilde{Q}\lambda + \tilde{S} = 0,$

where

1. $T = (2 + \alpha + \beta)Rx^* - (1 + R),$
2. $\tilde{P} = \frac{3T}{1+k},$
3. $\tilde{Q} = \frac{3T^2 - 3\alpha\beta R^2 x^*}{(1+k)^2},$
4. $\tilde{S} = \frac{T^3 - \alpha^3 R^3 x^{*3} - \beta^3 R^3 x^{*3} - 3T\alpha\beta R^2 x^{*2}}{(1+k)^3}.$

According to the Jury criterion, the following Jury table is obtained.

Table 1

The Jury table of $\Delta(\lambda)$

Rows	λ^0	λ^1	λ^2	λ^3
1	$\tilde{S}$	$\tilde{Q}$	$\tilde{P}$	1
2	1	$\tilde{P}$	$\tilde{Q}$	$\tilde{S}$
3	$\tilde{S}^2 - 1$	$\tilde{S}\tilde{Q} - \tilde{P}$	$\tilde{S}\tilde{P} - \tilde{Q}$	0

In order to determine that the controlled system (4) is stable on fixed point $\left(\frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}\right)$, three conditions need to be met as follow:

1. $\Delta(1) > 0.\ \Delta(1) = 1 + \tilde{P} + \tilde{Q} + \tilde{S} > 0,$
2. $1 - \tilde{P} + \tilde{Q} - \tilde{S} > 0,$
3. $|\tilde{S}| < 1$ and $|\tilde{S}^2 - 1| > |\tilde{S}\tilde{P} - \tilde{Q}|.$

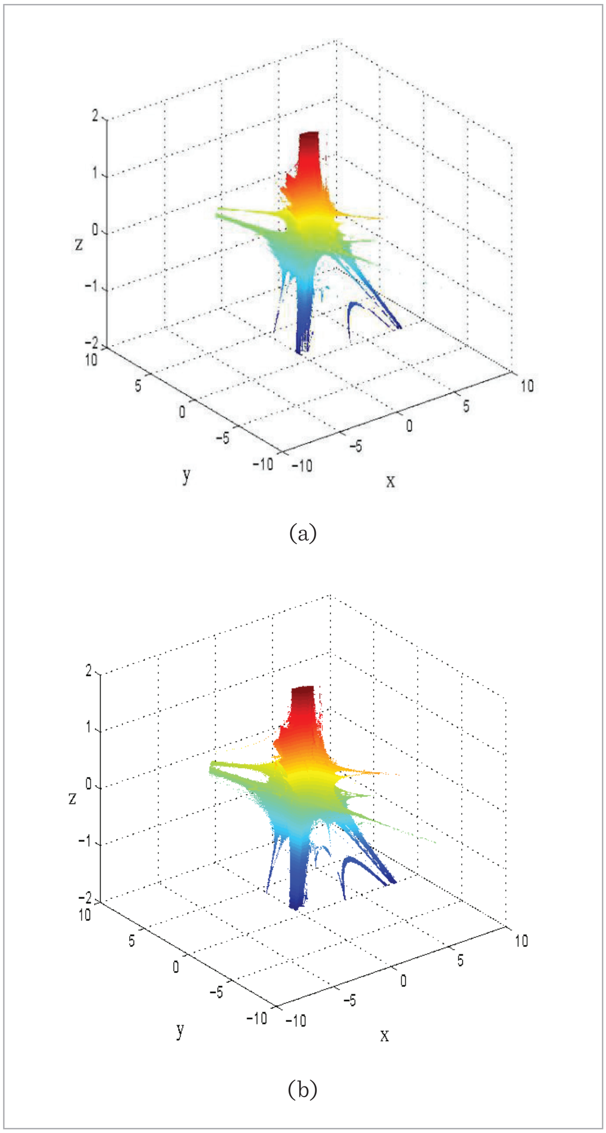
Take the parameters as R = 1, α = 0.4745, β = 3. The original Julia set of system (2) is shown in Figure 1. In Figure 1(b), the filled K(F) with $z_0 = 0$ is illustrated. Actually this can be regarded as a reduced case of system (2) when one of the third species is nonexistence. The trajectories of the two signed points $A \in K(F)$ and $B \notin K(F)$ are illustrated in Figures 2-3, in which we can see that the trajectories remain closed before 90 steps nevertheless B's trajectory begins to diverge when the iteration continues.

For the controlled system (4), the control parameters k should satisfy that

$k \in \{k | k < -2.2631 \cup k > 0.2631\}$. The controlled Julia sets and its slices are illustrated in Figures 4-5. The simulations shows that $J(F_c)$ expands with the values of the control parameter k increases in two directions of the number axis. In other words, more points come into the filled Julia set around the stable coexistence fixed point.

Figure 4

The controlled Julia sets $J(F_c)$ with parameters: (a) k = 0.27; (b) k = 0.4



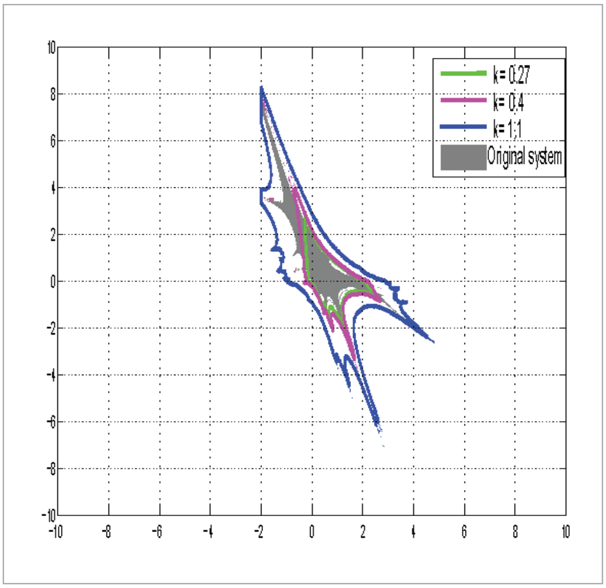
(a)

(b)

For both two species and multi species ecosystems, the complex coupling behavior among species are both complicated. For the previous works [21, 27, 35, 36], the range of control parameter k is obtained by

Figure 5

The 2D slices of controlled Julia sets J(Fc) with different parameters: (a) k >0.2631; (b) k< −2.2631



analyzing the stability of fixed point. There are still some deficiencies about the boundedness analysis of the original Julia set, and the boundedness change with different parameters k.

Note that in recent work [30], the authors proposed a preliminary frame to address the boundedness analysis problems for a kind of weakly chain coupled Logistic type map. Figure 6 [7] illustrates the complex symmetrical coupled predatory behavior of system (4). What we want is to apply the ideas in reference [30] to provide a preliminary study on the boundedness analysis of Julia sets in system (4). Now we reduce system (4) into a nutrient-phytoplankton-zooplankton version which contains a chain coupled predatory behavior:

$$F_{NPZ}\begin{cases} N_{n+1} = N_n + RN_n(1 - N_n - \alpha P_n - \beta Z_n), \\ P_{n+1} = P_n + RP_n(1 - \alpha Z_n - P_n), \\ Z_{n+1} = Z_n + RZ_n(1 - Z_n), \end{cases} \tag{5}$$

where R, α, β have the same meaning with system (4), N, P, Z represent the nutrient, phytoplankton and zooplankton respectively, which means that Z is a self-organizing species without predator. The chain coupled predatory behavior of system (5) is shown in Figure 7.

Figure 6

The symmetrical loop coupled predatory behavior of system (4). This figure is referenced from [7]

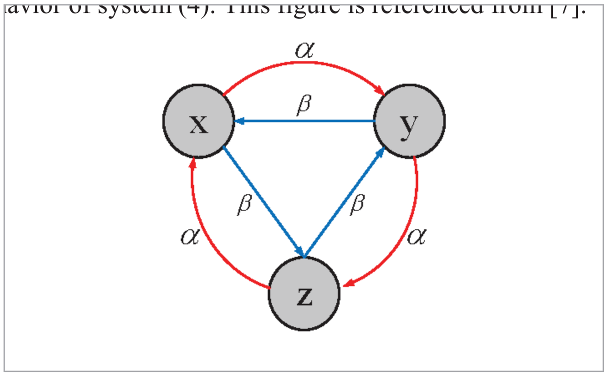
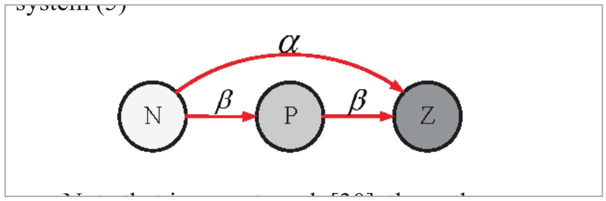


Figure 7

The chain coupled predatory behavior of system (5)



The upper bound of $J(F_{NPZ})$ system is given by the following theorem:

Theorem 1.

$J(F_{NZP}) \subset \{(N_0, P_0, Z_0) \mid |i_0| < B_i, i = N, P, Z\}$

in which $B_Z = \frac{|R+1|}{|R|}$, $B_P = \frac{|R+1|+|R\alpha B_Z|}{|R|}$ and

$B_N = \frac{|R+1|+|R\alpha B_P|+|R\beta B_Z|}{|R|}$.

proof. For the zooplankton without predator, if $|Z_0| > \frac{|R+1|}{|R|}$,one can get:

$|Z_1| \geq |Z_0|^2 - R - |Z_0|(|1+R|)$.

Then $\exists\epsilon$ such that

$\lim_{n\to\infty} |Z_{n+1}| = \lim_{n\to\infty}(1+\epsilon)^n |Z_0| \to \infty$ .So $|Z_0| < \frac{|R+1|}{|R|}$ is a necessary condition to make $J(F_{NPZ})$ bounded. If $|P_0| \geq B_p$, then one can get:

$$\begin{aligned} |P_1| &= |(1+R)P_0 - RP_0^2 - R\alpha Z_0 P_0| \\ &\geq |R||P_0|^2 - (|1+R|P_0 + |R\alpha||Z_0||P_0|) \\ &\geq |R||P_0|^2 - (|1+R| + |R\alpha B_Z|)P_0 \end{aligned}$$

Then $\exists\epsilon$ such that $\lim_{n\to\infty}|P_{n+1}| = \lim_{n\to\infty}(1+\epsilon)^n |P_0| \to \infty$. So it is clear that $|P_0| < B_P$ is a necessary condition to make $J(F_{NPZ})$ bounded. Similarly, if $|N_0| > B_N$, one can get:

$$|N_1| = |(1+R)N_0 - RN_0^2 - R\alpha P_0 N_0|$$
$$\geq |R||N_0|^2 - |N_0|(|1+R| + |R\alpha B_P| + |R\beta B_Z|)$$

Then $\exists\epsilon$ such that

$\lim_{n\to\infty}|N_{n+1}| = \lim_{n\to\infty}(1+\epsilon)^n |N_0| \to \infty$. So it is clear that $|N_0| < B_N$ is also a necessary condition to make $J(F_{NPZ})$ bounded.

The boundedness analysis of weakly coupled system (5) can be treated as an attempt to explore the system with more strongly coupled predatory behavior.

2.Synchronization of Julia Sets

In the nonlinear science area, synchronization [11] is regarded as another control method which aims to make the dynamical behavior of controlled system same to the ideal one. The synchronization of Julia sets can be realized by adding some coupling item to associate the controlled system with an ideal one. Consider the following system (5) with the same structure but different parameters as system (4):

$$\tilde{F}: \begin{cases} u_{n+1} = u_n + \tilde{R}u_n(1 - u_n - \tilde{\alpha}v_n - \tilde{\beta}w_n), \\ v_{n+1} = v_n + \tilde{R}v_n(1 - \tilde{\beta}u_n - v_n - \tilde{\alpha}w_n), \\ w_{n+1} = w_n + \tilde{R}w_n(1 - \tilde{\alpha}u_n - \tilde{\beta}v_n - w_n). \end{cases} \quad (6)$$

In order to associate system (6) with the ideal system (4), we introduce two coupling items into it to get

$$\widetilde{F^O}: \begin{cases} u_{n+1} = 2u_n - u_n^2 - \tilde{\alpha}u_n v_n - \tilde{\beta}u_n w_n - L_1(u_n - x_n) \\ -\widetilde{L_1}(\beta x_n z_n + \alpha x_n y_n + x^2 - \tilde{\beta}u_n v_n - \tilde{\alpha}u_n v_n - u_n^2) \\ v_{n+1} = 2v_n - v_n^2 - \tilde{\alpha}u_n w_n - \tilde{\beta}u_n v_n - L_2(v_n - y_n) \\ -\widetilde{L_2}(\beta x_n y_n + \alpha y_n z_n + y^2 - \tilde{\beta}u_n v_n - \tilde{\alpha}v_n w_n - v_n^2) \\ w_{n+1} = 2w_n - w_n^2 - \tilde{\alpha}u_n w_n - \tilde{\beta}v_n w_n - L_3(w_n - z_n) \\ -\widetilde{L_3}(\beta y_n z_n + \alpha x_n z_n + z^2 - \tilde{\beta}v_n w_n - \tilde{\alpha}u_n w_n - w_n^2) \end{cases} \quad (7)$$

The Julia sets of system (7) and system (4) are respectively denoted as $J(\widetilde{F^O})$ and $J(F)$. Give the coupling parameter L_j and $\widetilde{L_j}$ (j = 1, 2, 3) a uniform expression L and then we have the following lemma and theorem.

Lemma 1. [40].The synchronization between $J(\widetilde{F^O})$ and $J(F)$ is realized if there exit some L_0 such that

$$\lim_{L\to L_0}\left(J(\tilde{F}^O)\cup J(F) - J(\tilde{F}^O)\cap J(F)\right) = \varnothing.$$

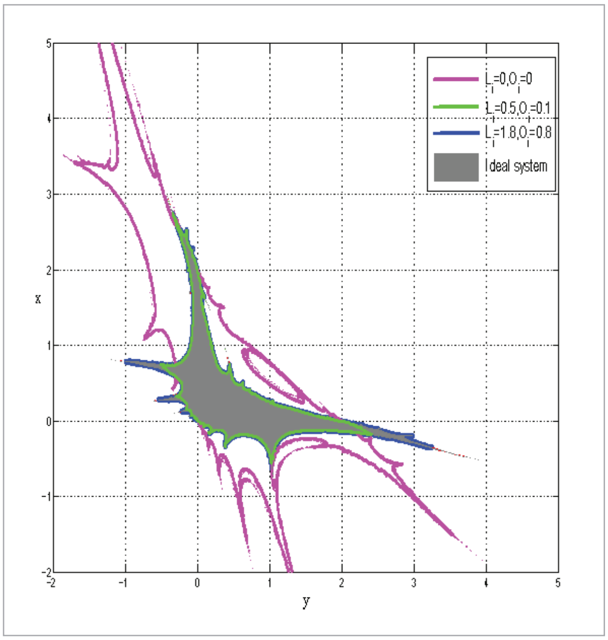
Lemma 2. $J(\tilde{F})$ and $J(F)$ are synchronized when $L_i \to 1$, $\tilde{L}_i \to 2$.

proof. It is clear that Julia set is obtained according to iteration of points within a bounded region which is denoted as Ω. Since $\psi_0 \notin J(F)$ if there exists n_0 such that $F^{n_0}(\psi_0)$ escape from region Ω. So in the synchronization process, only the points whose trajectories are inside Ω are needed to be considered. Therefore, $\exists N_1$ such that $|\beta||x_n z_n| + |\tilde{\beta}||u_n w_n| < N_1$ $|\alpha||x_n y_n| + |\alpha||u_n v_n| < N_1$, and $|x_n^2| + |u_n^2| < N_1$.

Then, we have

$$|u_{n+1} - x_{n+1}| \leq |Z - L_1||u_n - x_n|$$
$$+|1 - \widetilde{L_1}||\alpha x_n y_n - \tilde{\alpha}x_n y_n|$$
$$+|1 - \widetilde{L_1}||\beta x_n y_n - \tilde{\beta}u_n w_n|$$
$$+|1 - \widetilde{L_1}||x_n^2 - u_n^2|$$
$$\leq |2 - L_1||u_n - x_n|$$
$$+|1 - \widetilde{L_1}|(|\beta||x_n z_n| + |\tilde{\beta}||u_n w_n|)$$
$$+|1 - \widetilde{L_1}|(|\alpha||x_n y_n| + |\tilde{\alpha}||u_n v_n|)$$
$$+|1 - \widetilde{L_1}|(x_n^2 + u_n^2)$$
$$\leq |2 - L_1||u_n - x_n| + 3|1 - \widetilde{L_1}|N_1$$
$$\leq |2 - L_1|(|2 - L_1||u_{n-1} - x_{n-1}|$$
$$+3|1 - \widetilde{L_1}|N_1) + 3|1 - \widetilde{L_1}|N_1$$
$$\leq |2 - L_1|^2|u_{n-1} - x_{n-1}|$$
$$+3|1 - \widetilde{L_1}|N_1(1 + |2 - L_1|)$$
$$\leq \cdots \ldots$$
$$\leq |2 - L_1|^n|u_1 - x_1|$$
$$+3|1 - \widetilde{L_1}|N_1(1 + |2 - L_1|$$
$$+|2 - L_1|^2 + \cdots + |2 - L_1|^n)$$
$$= |2 - L_1|^n|u_1 - x_1|$$
$$+3|1 - \widetilde{L_1}|N_1\frac{1 - |2 - L_1|^n}{1 - |2 - L_1|}.$$

Figure 8
The synchronization of $J(\widetilde{F^O})$ with different L



The limitation of the right side satisfies that $L_{2,3} \to 2$, $\widetilde{L_{2,3}} \to 1$, the trajectories of u_n and x_n is synchronized. Similarly, when $L_{2,3} \to 2$, $\widetilde{L_{2,3}} \to 1$, the trajectories of v_n and y_n, w_n and z_n are synchronized.

Fig. 8 presents the ideal Julia set J(F) and the changing process of Julia sets from the coupling system (7) with different parameters L.

3. Conclusion and Further Discussions

The Lotka-Volterra system is considered as the most basic theoretical model in the research area of population biology. Chaos and bifurcations are the main contents in the nonlinear dynamical analysis about Lotka-Volterra system. Note that fractals is one of the main frontiers of nonlinear sciences. Several recent studies have extended the dynamical analysis of Lotka-Volterra system into the category of fractals. Nevertheless, these fractals investigations of ecosystems mainly focused on 2D systems, and not sufficiently studied the topological boundedness properties. Considering that multi species systems have better explanation to the actual biological processes, this paper extends these ecosystem fractals researches into three-species case by considering a kind of *GLV* model. The definition of Julia sets for the *GLV* model is given which helps us to explore the initial species states. Then we apply the giant control method which contains both giant parameter and the state feedback to achieve the Julia sets' control. A preliminary study on the boundedness analysis of Julia sets on a kind of weakly coupled nutrient-phytoplankton-zooplankton model is also proposed. At last, the synchronization of Julia sets is considered to realize the coupling of two different systems. Some further discussions of this work can be summarized as follows:

1 Note that for some *GLV* systems with more general structure, such as the system whose predatory behavior lose the symmetrical coupled structure [2], or system with more species [16], the complicated coupled behavior increase the difficulty in analyzing the topological properties of Julia set.

2 Moreover, it seems that there exist some stripe and spot patterns [31, 32] in the fractals structure of *GLV* system (see Figure 1). Thus in the future work, the relationship between fractals and patterns in ecosystems can be investigated.

3 At last, further study combining fractals and ecosystems should pay much more attentions on its practical application. For example, could the evolutionary process of the actual species provide effective data support to identify a corresponding fractals system? If so, which actual generation with stable evolution is proper to be taken as the escape times of the Julia set?

Thus the idea and study enclosed in this paper could be further expanded and deepened in the cases of systems with more general form and potential applications. Based on the analysis above, there is still plenty of work that needs further investigation.

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