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The Use of the Lambert Function Method for Analysis of a Control System with Delays

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Abstract. The mathematical model of the mutual synchronization system with complete graph structure, composed of n ($n \in N$) oscillators, is investigated. This mathematical model is defined by the matrix differential equation with delayed argument. The solution of the matrix differential equation with delayed argument is obtained by applying the Lambert W function method. On the base of this solution, the step responses matrix of the synchronization system is defined and the transients in the system are investigated. The results of calculations, received by the Lambert function method, the dde23 method in Matlab and the exact method of consequent integration, are compared.

Keywords: synchronization system; differential equations; delayed argument; Lambert function.

1. Introduction

Control systems and engineering techniques have become an integral part of modern technology. These systems are often components added to other complex systems to increase their functionality or to meet the set of design criteria. Usually they are being investigated by applying their mathematical models. More exact analysis of the systems demands the use of the more complicated mathematical models. Often the delays of the signals, transferred along the control system, must be included into these models. The delays make the investigation of the model more cumbersome. Usually investigation of such model demands solution of delay differential equation. The principal difficulty in solving delay differential equation lies in its special transcendental character. The characteristic equation of linear delay differential equation is transcendental and has infinite number of roots. For solution of this characteristic equation in the present work we use a method based on the application of Lambert W functions. The Lambert W functions for analytic investigations of various dynamical systems with delays were applied by several authors [1–5]. In [1, 2], the new analytic approach to obtain the complete solution for systems of delay differential equations based on the concept of Lambert W function is presented. In [3], Yi et al. have considered the problem of feedback controller design via eigenvalue assignment for systems of linear delay differential equations using Lambert W function method. In [4], the approach of using the Lambert W functions to time domain analysis of a class of fractional order time delay systems is extended. In [5], a survey on analysis of delay systems via Lambert W function is given. In all these works the systems of differential equations of order not greater then third were investigated by applying the Lambert W function method.

In the present work, we apply Lambert W function method to investigate synchronization systems, described by linear systems of delay differential equations up to fifteenth order. The relative errors of the obtained results are evaluated using the solutions obtained by the exact method of consequent integration.

2. Formulation of the problem

In the presented work, the multidimensional control system with delays and with structure of complete graph is investigated. The mathematical model of this system is the matrix differential equation with delayed argument [6–9]

$$x'(t)+B_1x(t)+B_2x(t-\tau)=z(t), \qquad (1)$$
$$x(t)=\phi(t), \quad t\in[-\tau,0],$$

where $x(t)=\left(x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)\right)^T$ is the desired vector function, T(here and in what follows) denotes the operation of transposition, τ is a constant time delay, $\phi(t)$ is a vector–valued preshape (initial) function, $z(t)$ is a free term (continuous function depending on the initial conditions), κ is a coefficient, B_1 and B_2 are $n\times n$ ($n\in N$) numerical matrices $\left(B_1, B_2\in R^{n\times n}\right)$,

$$B_1=\kappa E, \qquad (2)$$

$$B_2=\frac{\kappa}{n-1}B, \qquad (3)$$

$$B=\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1\\ 1 & 0 & 1 & \dots & 1 & 1\\ 1 & 1 & 0 & \dots & 1 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad (4)$$

($E\in R^{n\times n}$ is the identity matrix, matrix $B\in R^{n\times n}$ outlines the structure of the internal links of the system).

As an example of a control system, described by the equation (1), the mutual synchronization system of the communication network, having structure of the complete graph and composed of n oscillators, can be pointed out [8] (in Fig. 1 the scheme of the internal links of the system, composed of 5 oscillators, is presented). In this case, the symbol $x_i(t)$ in (1) stands for the phase of the i–th oscillator.

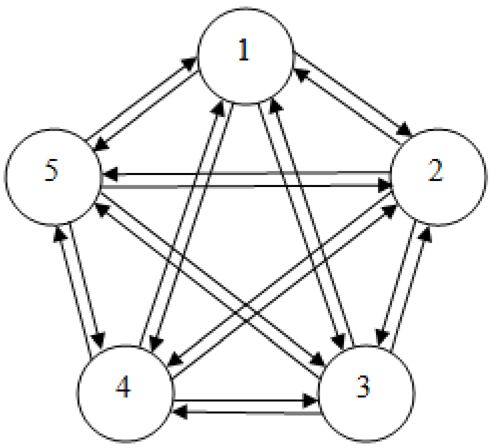


Figure 1. The scheme of internal links of the system, when $n=5$

We shall assume that

$$x_i(t)=\begin{cases}\int_0^t f_i(\xi)d\xi+x_{0i}, & \text{if } t>0,\\ f_{0i}t+x_{0i}, & \text{if } \quad t\le 0;\end{cases} \qquad (5)$$

here $x_{0i}=x_i(0)$ $(i=\overline{1,n})$ is the initial phase of the i–th oscillator's oscillation, $f_i(t)$ is the frequency of the i–th oscillator, f_{0i} is the own frequency (the frequency of the i–th oscillator when the control signal is disconnected). The meaning of (5) is the following: the control signal to the i–th oscillator at time $t=0$ is connected up. Before this time moment the i–th oscillator works with its own frequency f_{0i}. Taking into account (5), we get the following expressions for the initial vector function $\phi(t)$ and the free vector $z(t)$ of (1):

$$\phi(t)=(\phi_1(t) \quad \phi_2(t) \quad \cdots \quad \phi_n(t))^T, \qquad (6)$$

$$\phi_i(t)=f_{0i}t+x_{0i}, i=\overline{1,n} \quad ,t\in[-\tau,0], \qquad (7)$$

$$z(t)=(f_{01} \quad f_{02} \quad \cdots \quad f_{0n})^T. \qquad (8)$$

3. Solution of the matrix differential equation with delayed argument

If to apply the Lambert W function method (see [10], p. 23), the solution of (1) on the interval $[0,+\infty)$ can be expressed as follows:

$$x(t)=\sum_{k=-\infty}^{\infty}e^{S_k t}C_k+\int_0^t\sum_{k=-\infty}^{\infty}e^{S_k(t-\xi)}C_k' z(\xi)d\xi=$$
$$=\lim_{N\to\infty}\sum_{k=-N}^{N}e^{S_k t}C_k+$$
$$+\int_0^t\left(\lim_{N\to\infty}\sum_{k=-N}^{N}e^{S_k(t-\xi)}C_k' z(\xi)\right)d\xi; \qquad (9)$$

here C_k is a $n\times 1$ coefficient matrix–column computed from the given preshape function $x(t)=\phi(t)$, which is an initial state of delay differential equation (1) for $t\in[-\tau,0]$, and C_k' is a $n\times n$ coefficient matrix computed from the given free term $z(t)$ of the matrix differential equation (1) (procedures of calculation of these matrices are explained in [1], p. 222 and [2], p. 2434). In the computations of the solution $x(t)$ we shall use the approximate expression, obtained from (9) at fixed and finite N:

$$x(t)=\sum_{k=-N}^{N}e^{S_k t}C_k++\int_0^t\left(\sum_{k=-N}^{N}e^{S_k(t-\xi)}C_k' z(\xi)\right)d\xi. \qquad (10)$$

4. The step responses matrix of the system

A good representation about transients in the system can be obtained by determining its responses to perturbations having form of the unit jump [6]. The response of the i–th oscillator's oscillation phase to a unit jump in the j–th oscillator's oscillation phase we shall call the step response $h_{ij}(t)$. The set of the step responses $h_{ij}(t)\left(i,\, j=\overline{1,n}\right)$ forms $n \times n$ matrix $h(t)=\left(h_{ij}(t)\right)$ (the step responses matrix of the synchronization system). We shall find the matrix $h(t)$.

When the increment of the phase of the j–th oscillator takes form of the unit jump, the increment of the free term of the equation (1) can be expressed as follows

$$\Delta z(t)=\delta(t) I^{(j)}; \qquad (11)$$

here $I^{(j)}$ is the matrix–column all entries of which are zeros except the j–th element, which is equal to 1, $\delta(t)$ is the Dirac delta function. Taking this into account and using (1), we get the following differential equation for step responses $h_{ij}(t)\left(i,\, j=\overline{1,n}\right)$ of the system:

$$h_j'(t)+B_1 h_j(t)+B_2 h_j(t-\tau)=\delta(t) I^{(j)}, \qquad (12)$$

$$j=\overline{1,n},$$

$$h_j(t)=0, \quad t \in [-\tau, 0];$$

here $h_j(t)=\left(h_{1j}(t) \quad h_{2j}(t) \quad \cdots \quad h_{nj}(t)\right)^T$ is the j–th column of the step responses matrix $h(t)$, matrices B_1 and B_2 are defined by (2) and (3), respectively.

Firstly, we shall find the solution of (12) on the interval $[0, \tau]$.

Column–vector $h_j(t-\tau)$ is a zero column–vector on the interval $[0,\tau]$ due to the initial conditions (see (12)). Taking this into account on the interval $[0,\tau]$, we get the following system of differential equations for the step responses $h_{ij}(t),\ i, j=\overline{1,n}$:

$$\left(h_{ij}(t)\right)' + \kappa h_{ij}(t)=\begin{cases}\delta(t), & \text{if } i=j,\\ 0, & \text{if } i \neq j.\end{cases} \qquad (13)$$

Solution of (13) is the set of functions: $h_{ij}(t)=0$ if $i \neq j$ and $h_{ij}(t)=e^{-\kappa t} 1(t)$ if $i=j$ ($i, j=\overline{1,n}$); here $1(t)$ is the Heaviside step function.

Using the solution of (13), the differential equation (12) on the interval $[\tau,+\infty)$ can be presented as homogeneous matrix delay–differential equation

$$h_j'(t)+B_1 h_j(t)+B_2 h_j(t-\tau)=0, \quad j=\overline{1,n},$$

$$h_j(t)=\left(h_{1j}(t) h_{2j}(t) \ldots h_{nj}(t)\right)^T=\varphi_j(t), \qquad (14)$$

$$t \in [0, \tau];$$

here $\varphi_j(t)$ is the preshape vector–function. The entries of the vector–function $\varphi_j(t)$ assume the following values:

$$\varphi_{ij}(t)=\left(\varphi_j(t)\right)_i=\begin{cases}e^{-\kappa t} 1(t), & \text{if } i=j,\\ 0, & \text{if } i \neq j.\end{cases} \qquad (15)$$

Applying the Lambert function method, the solution of (14) on the interval $[\tau,+\infty)$ can be expressed as follows [see [2], p. 2434]:

$$h_j(t)=\sum_{k=-\infty}^{\infty} e^{S_k t} C_k(j)=$$

$$=\lim_{N \to \infty} \sum_{k=-N}^{N} e^{S_k t} C_k(j),\ j=\overline{1,n}, \quad t \in [\tau,+\infty); \qquad (16)$$

here

$$S_k=\frac{1}{\tau} W_k\left(-B_2 T e^{B_1 \tau}\right)-B_1,$$

$W_k\left(-B_2 T e^{B_1 \tau}\right)$ is the value of the k–th branch $W_k(H)$ of the matrix Lambert function $W(H)$ at $H=-B_2 T e^{B_1 \tau}$, $C_k(j)$ are the complex–valued vectors corresponding to the preshape vector–function $\varphi_j(t)$ (see (15)). The algorithms for finding $C_k(j)$ and W_k are explained in [2, p. 2434–2435]. From (16) the approximate expression for $h_j(t)$ follows:

$$h_j(t)=\sum_{k=-N}^{N} e^{S_k t} C_k(j), \quad j=\overline{1,n}, \qquad (17)$$

$$t \in [\tau,+\infty);$$

here N is a sufficiently large natural number.

5. Investigating of stability

Analyzing the distribution of roots of the transcendental characteristic equation of delay system we can obtain information about its stability. Let's write down the characteristic equation of investigated system and find out the closed form solution of it.

Let's write down homogeneous differential equation corresponding to (1):

$$x'(t)+B_1 x(t)+B_2 x(t-\tau)=0,$$

$$x(t)=\phi(t), \quad t \in[-\tau, 0]. \qquad (18)$$

Assuming that a solution of homogeneous differential equation is a vector function

$$x(t)=e^{S t} C \qquad (19)$$

and substituting it into (18), we get the transcendental characteristic equation

$$S+B_1+B_2 e^{-S \tau}=0 \qquad (20)$$

(here S is $n \times n$ numerical matrix, C is a nonzero numerical vector with n entries; the entries of S and C are some complex numbers). Function (19) will be a solution of the homogeneous differential equation (18) if matrix S in its expression will be a root of the transcendental characteristic equation (20). We shall find the closed form expression for roots of (20).

Multiplying both sides of (20) by $e^{S\tau}$, we get

$$(S+B_1)e^{-S\tau}=-B_2. \tag{21}$$

Performing further transformations, we multiply both sides of (21) by $\tau e^{B_1\tau}$. This yields

$$(S+B_1)\tau e^{-S\tau}e^{B_1\tau}=-B_2\tau e^{B_1\tau}.$$

Recall that the matrices B_1 and B_2 in equation (18) commute. Then S and B_1 will commute as well (see, for example, [10], p. 119). Taking this into account, we can write

$$\tau(S+B_1)e^{\tau(S+B_1)}=-B_2\tau e^{B_1\tau}. \tag{22}$$

We know [12] that the Lambert W function is a function satisfying equality

$$W(z)e^{W(z)}=z.$$

So, we can write

$$W(-B_2\tau e^{B_1\tau})e^{W(-B_2\tau e^{B_1\tau})}=-B_2\tau e^{B_1\tau}. \tag{23}$$

Equating (22) and (23), we obtain

$$\tau(S+B_1)=W(-B_2\tau e^{B_1\tau}).$$

From this equality, we get the closed form expression for the roots of (20):

$$S=\frac{1}{\tau}W(-B_2\tau e^{B_1\tau})-B_1.$$

Since the Lambert W function has an infinite number of branches, the matrix transcendental characteristic equation (20) will have an infinite number of roots, which can be expressed as follows:

$$S_k=\frac{1}{\tau}W_k(-B_2\tau e^{B_1\tau})-B_1, \tag{24}$$

$$k=0,\pm1,\pm2,\ldots.$$

If matrix $H=-B_2\tau e^{B_1\tau}$ is diagonalizable, then we compute the eigenvalues $\Lambda_i,\quad i=\overline{1,n}$ of H and the corresponding eigenvector matrix$V$. To each branch k $(k=-\infty,\ldots,-1,0,1,\ldots,+\infty)$ of the Lambert W function, we get:

$$W_k(H)=V\mathrm{diag}\big(W_k(\Lambda_1),W_k(\Lambda_2),\ldots,W_k(\Lambda_n)\big)V^{-1}.$$

If H is not diagonalizable, then $W_k(H)$ has more complicated structure (see, for example, [17], p. 2125).

Having found matrix S_k ($k=0,\pm1,\pm2,\ldots$), we calculate its eigenvalues $\lambda_{k,i}\,(i=\overline{1,n})$. Analysis of distribution of these eigenvalues on the complex plane provides information about stability of the system (the system can be asymptotically stable, unstable or marginally stable [16]).

6. Comparing the Lambert function method with the exact method of consequent integration

The solutions of matrix delay differential equations (1) and (14) are presented by the infinite functional series (see (9) and (16)), which determines the exact solutions. In the real calculations we apply the approximate formulas (10) and (17), obtained from (9) and (16) with finite N ($2N+1$ indicates the number of branches of the Lambert W function, which are used in calculation of the solutions).

We shall investigate the rate of convergence of the approximate solutions of matrix delay–differential equation (14) to its exact solution with increasing N. For this purpose, we shall apply the exact expressions found by the method of consequent integration (method of “steps”) [8, 11].

We present the solution of (1), applying the Laplace transform, as follows [8]:

$$x(t)\div\sum_{l=1}^{d}(A^{-1}B_2e^{-pt})^l A^{-1}Z(p), \tag{25}$$

$$0<t<(d+1)\tau;$$

here $A=pE-B_1=(p+\kappa)E,\quad Z(p)\div z(t),\quad Z(p)$ is the Laplace transform of the vector function $z(t)$ (sign $\div$ links function with its Laplace transform), $d=0,1,2,\ldots$. Taking into account (2)–(4), we write

$$x(t)\div\sum_{l=0}^{d}\left(\frac{\kappa}{n-1}\right)^l\frac{1}{(p+\kappa)^{l+1}}e^{-pl\tau}B^lZ(p),$$

$$0<t<(d+1)\tau$$

and

$$x(t)=\sum_{l=0}^{d}\left(\frac{\kappa}{n-1}\right)^l B^lL^{-1}\left(\frac{1}{(p+\kappa)^{l+1}}e^{-pl\tau}Z(p)\right), \tag{26}$$

$$0<t<(d+1)\tau;$$

here $L^{-1}(F(p))$ is the inverse Laplace transform of $F(p)$.

Let’s write down the step responses matrix of the system. Using (1), (12) and (25), we obtain

$$h(t)=(h_{ij}(t))\div\sum_{l=0}^{d}\left(\frac{\kappa}{n-1}\right)^l\frac{1}{(p+\kappa)^{l+1}}e^{-pl\tau}B^l,$$

$$0<t<(d+1)\tau.$$

The inverse Laplace transform, applied to the right hand side of the latter expression, gives

$$h_{ij}(t) = \sum_{l=0}^{d} \left(\frac{\kappa}{n-1} \right)^{l} \{B^{l}\}_{ij} \frac{(t-l\tau)^{l}}{l!} \times$$
$$\times e^{-\kappa(t-l\tau)} 1(t-l\tau), \qquad (27)$$
$$0 < t < (d+1)\tau;$$

here $\{B^{l}\}_{ij}$ is the ij–th entry of the matrix B^{l}. Note, that (26) and (27) represent the exact expressions of the solutions on the interval $(0,(d+1)\tau)$.

The step response $h_{11}(\kappa t)$ of mutual synchronization system with a complete graph structure, computed by Lambert W function method with different values of N and by the exact method of consequent integration, are presented in Fig. 2. In this figure the graphs of the solutions, obtained by dde23 program in MATLAB, for the sake of comparison, are presented, as well.

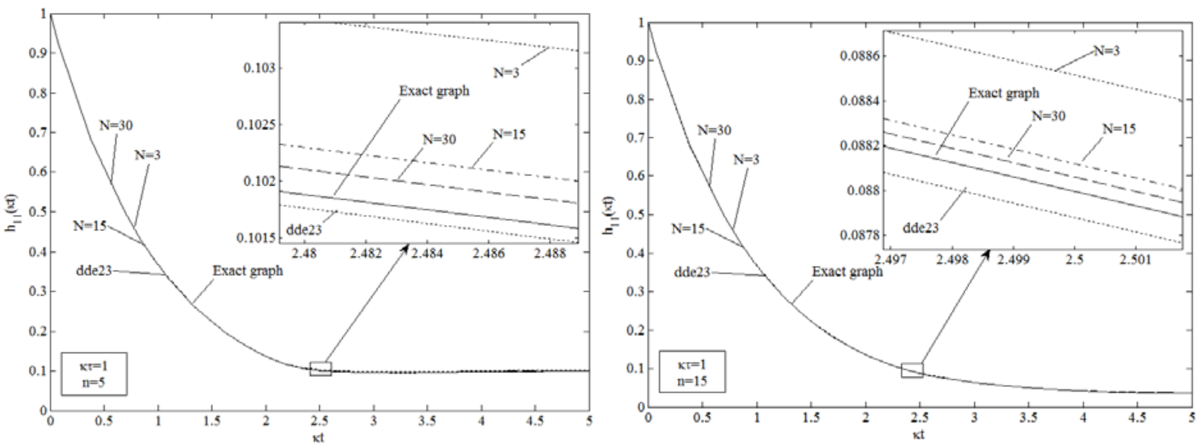
The relative errors δ obtained at $\kappa t = 2.5$ (at mid point of the interval $[0,5]$), using the Lambert W function method with different values of N and the numerical method based on the dde23 program in MATLAB, are presented in the Table 1, when $n = 5$, $\kappa = 1$ Hz, and in the Table 2, when $n = 15$, $\kappa = 1$ Hz. We can see from the tables, that, if τ is small ($\kappa t \ll 1$), then the results obtained by Lambert W function method are more accurate if to compare them with the corresponding results got by the dde23 program in MATLAB. With an increase of the number of oscillators n, the accuracy of the Lambert W function method has tendency to increase.

Table 1. The relative error δ when $n = 5$

$\kappa\tau$	LAMBERT				DDE23
	N				
	1	5	30	80	
0.01	0.0000027	0.0000023	0.000013	0.0000054	0.0016
0.1	0.00083	0.00057	0.00013	0.000051	0.00019
1	0.017	0.0039	0.00075	0.00029	0.00092

Table 2. The relative error δ when $n = 15$

$\kappa\tau$	LAMBERT				DDE23
	N				
	1	5	30	80	
0.01	0.000045	0.000034	0.000023	0.0000092	0.0093
0.1	0.0014	0.0010	0.00023	0.000092	0.0092
1	0.0370	0.0109	0.0022	0.000089	0.00092



(a) (b)

Figure 2. Graphs of the step response $h_{11}(\kappa t)$ calculated by three methods: 1) the Lambert W function method with different values of N, 2) numerical method using dde23 program in MATLAB, 3) exact method of consequent integration (method of "steps")

7. Numerical results

7.1. Transients

For the calculation of the phase differences $x_i(t) - x_j(t)$ and the step responses $h_{ij}(t)$, we have applied the formulas (10) and (17), respectively, with $N = 80$ (this means that we have used 161 branches of the Lambert W function in the computations). On the base of the results in Section 6, the graphs of the transients, presented below, are sufficiently accurate (in the presented figures these graphs practically coincide with the exact ones).

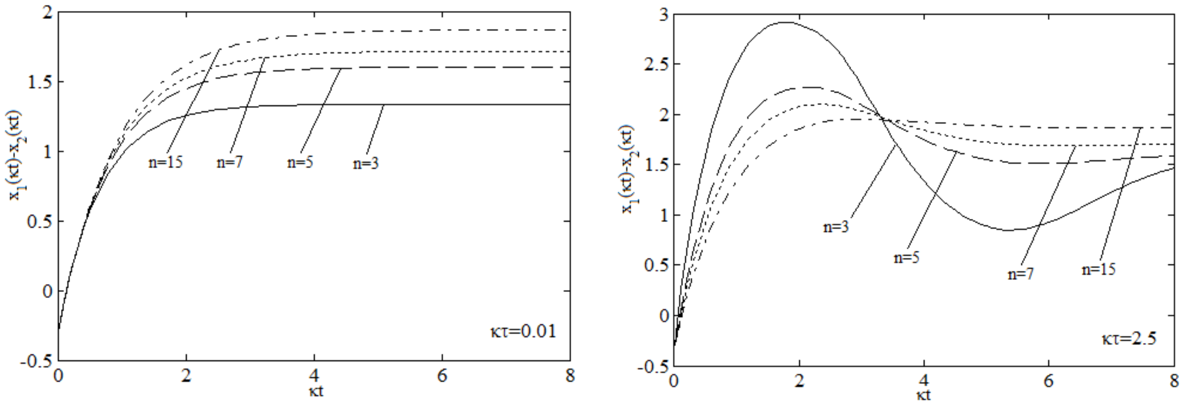
The calculations are performed assuming:

$$\frac{f_{i0}}{\kappa} = \begin{cases} 2003, & i = \overline{1,5}; \\ 2001, & i = \overline{6,10}; \\ 1999, & i = \overline{11,15} \end{cases}$$

and

$$x_{i0} = \begin{cases} 0.3, & i \text{ is } odd; \\ 0.6, & i \text{ is } even. \end{cases} \qquad (28)$$

In Fig. 3 the graphs of the phase difference $x_2(t) - x_1(t)$ are given at different values of $\kappa\tau$ and for different numbers of oscillators in the synchronization system with parameters of the system defined by (28). From the figure, we see that the character of the transients in the synchronization system crucially depends on n (the number of oscillators in the synchronization system). If $n = 3$, the transients get oscillatory features when $\kappa\tau \geq 1$, and with increase of $\kappa\tau$ the oscillatory features of the transients tend to increase. If $n = 15$, the transients in the system go without oscillatory features even when $\kappa\tau = 2.5$. With an increase of n and $\kappa\tau$ the duration of transients in the system changes insignificantly. In Fig. 4 the graphs of the phase differences $x_1(t) - x_j(t)$, $j = 4{,}10{,}14$ are presented.



(a) (b)

Figure 3. Graphs of the phase difference $x_1(t) - x_2(t)$

The graphs of some step responses are given in Fig. 5. The graphs presented on these figures show that the system under consideration is marginally stable since the step responses tend to positive finite values when κt tends to infinity (the characteristic equation of the system has simple zero root).

7.2. Stability

Analyzing the distribution of the eigenvalues of matrices S_k ($k = 0, \pm 1, \pm 2, \ldots$) (the roots of the characteristic equation (20)) on the complex plane one can make a conclusion about system's stability. In Fig. 6, the distribution of the eigenvalues $\lambda_{k,i}(i = \overline{1,5})$ of the matrices $S_k(k = 0{,}1{,}2)$ is presented on the complex plane for the case $n = 5$ (the eigenvalues of the matrices $S_k(k = 3{,}4{,}5,\ldots)$ are not shown here since they have greater in absolute value negative real parts and are located outside the drawing). From the figure it follows that the right most eigenvalue is $\lambda_{0,1}$. This eigenvalue is simple and has zero real part. This fact indicates that the system is marginally stable [16]. This conclusion coincides with the one obtained from the analysis of the graphs of step responses. In Fig. 7, the relation between real parts of eigenvalues $\lambda_{0,i}(i = \overline{1,5})$ of matrix S_0 and delay τ is presented.

Similar conclusions follow for other values of n.

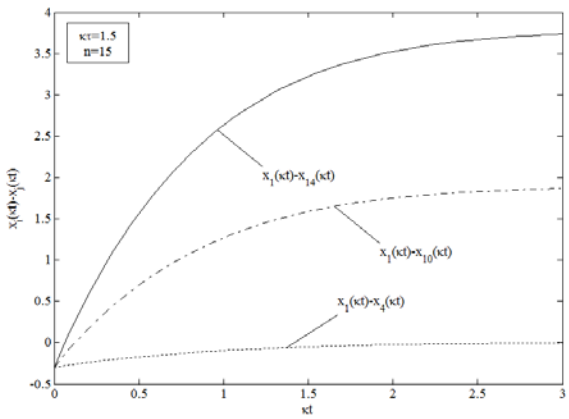
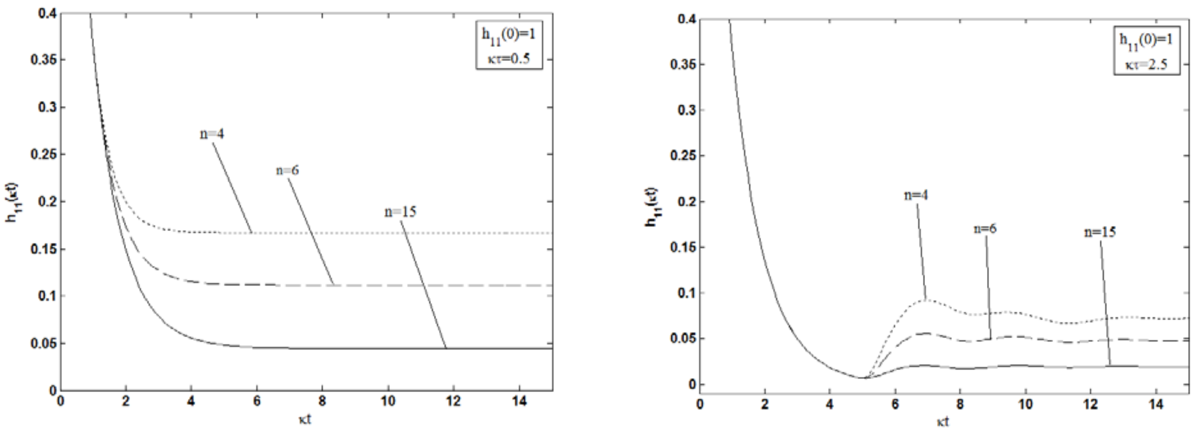


Figure 4. Graphs of the phase differences $x_1(t) - x_j(t)$

8. Conclusions

1. The Lambert function method is used for calculation of transients in the synchronization system, when the structure of the internal links in the system bear form of the complete graph. It is shown that using 161 branches of the Lambert W function (taking $N = 80$) in calculations of step responses $h_{ij}(\kappa t)$, the relative error is not greater than 0.001 for $\kappa t = 2.5$ and $n \leq 15$ (here n is the number of oscillators in the synchronization system).

2. The Lambert W function method has the advantage in comparison with a method of consequent integration (method of "steps"), as time of calculation of transients by this method does not depend on delay



(a) (b)

Figure 5. Graphs of the step response $h_{11}(\kappa t)$

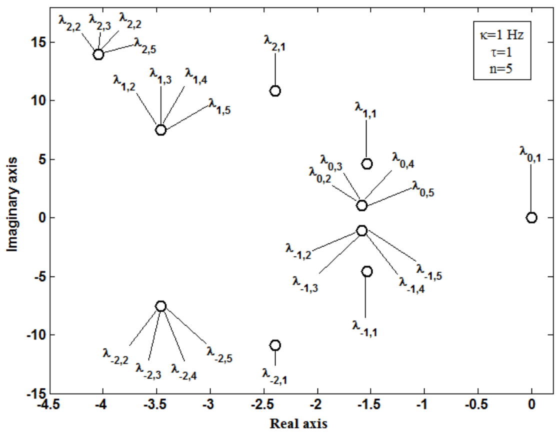


Figure 6. Distribution of eigenvalues $\lambda_{k,i}(i=\overline{1,5})$ of matrices $S_k(k=0,1,2)$ on the complex plane

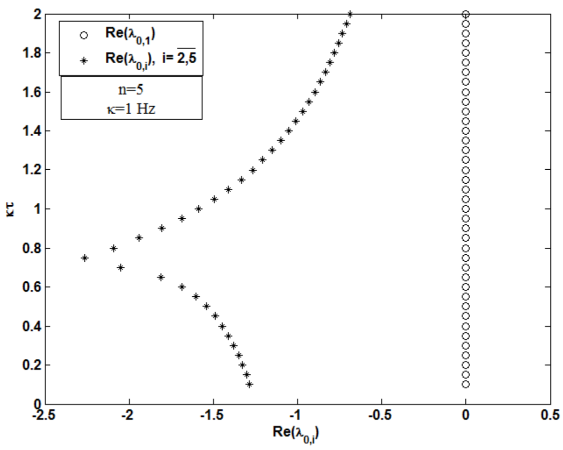


Figure 7. Relation between real parts of eigenvalues $\lambda_{0,i}(i=\overline{1,5})$ of matrix S_0 and delay τ

size, whereas time of calculation of transients by means of a method of consequent integration is in inverse proportion to the delay size.

3. The Lambert W function method has the advantage in comparison with a numerical method based on the application of dde23 program in MATLAB, if the product $\kappa\tau$ is small ($\kappa\tau \ll 1$).

4. The method of research of dynamics, used in the presented work, can also be applied to other control systems, described by the linear matrix differential equations with delayed arguments and with commuting coefficient matrices.

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