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On the Maximum and Minimum of Multivariate Pareto Random Variables

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Abstract. Aksomaitis and Burauskaite-Harju *[Information Technology and Control*, 38, 2009, 301-302] studied the distribution of max($X_1, X_2, ..., X_p$) when ($X_1, X_2, ..., X_p$) follows the multivariate normal distribution. Here, we study the moments of min($X_1, X_2, ..., X_p$) and max($X_1, X_2, ..., X_p$) when ($X_1, X_2, ..., X_p$) follows the most commonly known multivariate Pareto distribution. Multivariate Pareto distributions are most relevant for modeling extreme values.

Keywords: maximum; minimum; multivariate Pareto distribution.

1. Introduction

Let $(X_1, X_2, \ldots, X_p)$ be a continuous random vector with means $\mu_1, \mu_2, \ldots, \mu_p$ variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2$ and correlation coefficient ρ. The extreme values of $(X_1, X_2, \ldots, X_p)$ are $M = max(X_1, X_2, \ldots, X_p)$ and $m = min(X_1, X_2, \ldots, X_p)$. It is often of interest to know how $E(M)$ and $E(m)$ vary with respect to the means, variances and the correlation coefficient. For example, Smith and Sardeshmukh (2000) find that "The change of variance is associated both with altered skewness and a change in high and low extremes". Griffiths *et al.* (2005) use the change in mean temperature as a predictor of extreme temperature change in the Asia-Pacific region. Karl *et al.* (2008) observe that "A relatively small shift in the mean produces a larger change in the number of extremes for both temperature and precipitation". Nicholls (2008) argues the need for careful statistical analysis "since the likelihood of individual extremes, such as a late spring frost, could change due to changes in variability as well as changes in the mean climate". Burton and Allsop (2009) observe that "In recent years, there has been an increasing tendency to use mean speeds to predict extremes".

In the extreme value literature, only four papers have studied the distributions of M and m with respect to the means, variances and the correlation coefficient: Ker (2001), Lien (2005), Aksomaitis and Burauskaite-Harju (2009), and Hakamipour *et al.* (2011). Ker (2001) supposes that (X_1, X_2) has a bivariate normal distribution. Lien (2005) supposes that (X_1, X_2) has a bivariate lognormal distribution. Aksomaitis and Burauskaite-Harju (2009) suppose that $(X_1, X_2, \ldots, X_p)$ has a multivariate normal distribution. Hakamipour *et al.* (2011) suppose that $(X_1, X_2, \ldots, X_p)$ follows a multivariate Pareto distribution due to Muliere and Scarsini (1987). As we can see, the first two of these papers are limited to the case $p = 2$. The distributions considered by the first three papers (bivariate normal, bivariate lognormal, and multivariate normal) are not the most appropriate ones for modeling extreme values. Muliere and Scarsini's (1987) multivariate Pareto distribution, the distribution considered by the fourth paper, suffers from discontinuity and has limited applicability.

Bivariate and multivariate Pareto distributions have been the most popular distributions for modeling extremes. Among these, the most commonly known distribution due to Arnold (1983, Chapter 6) has the joint survivor function specified by

$$\bar{F}(x_1, x_2, \ldots, x_p) = \left[1 + \sum_{i=1}^{p} \frac{x_i}{\theta_i}\right]^{-\alpha} \quad (1.1)$$

for $x_i > 0$, $i = 1,2,\ldots,p$, $\theta_i > 0$, $i = 1,2,\ldots,p$ and $\alpha > 0$. This distribution has received widespread attention: Yeh (2004) studies extreme order statistics of (1.1); Li (2006) investigates tail dependence properties of (1.1); Cai and Tan (2007) propose (1.1) as a model for dependent risks; Vernic (2011) uses (1.1) to estimate tail conditional expectation, a popular measure of risk.

For the multivariate Pareto distribution given by (1.1), standard calculations show that

$$\mu_i = \frac{\theta_i}{\alpha-1}, \alpha > 1, \qquad (1.2)$$

$$\sigma_i^2 = \frac{\alpha\theta_i^2}{(\alpha-1)^2(\alpha-1)}, \ \alpha > 2 \qquad (1.3)$$

$$\rho = \frac{1}{\alpha} \qquad (1.4)$$

The aim of this short note is to study how $E(M)$ and $E(m)$ vary with respect to the means, variances and the correlation coefficient. The main results are Theorem 2.1, Theorem 2.2 and Theorem 2.3.

The level of mathematics required in Section 2 is not high, but not less elementary than the mathematics used in Ker (2001), Lien (2005), Aksomaitis and Burauskaite-Harju (2009), and Hakamipour *et al.* (2011). We feel that the given results are important because of the prominence of multivariate Pareto distributions and because of the prominence of (1.1) in modeling extreme values. Besides, the results known on this topic have so far been limited for the bivariate case or distributions which are not most significant in modeling extreme values.

2. Main results

The main results in this section need the following lemma.

Lemma 2.1. *Define*

$$I(k, a, \alpha) = \int_0^\infty x^{k-1}(1 + ax)^{-\alpha} dx.$$

Then,

$$I(k, a, \alpha) = a^{-k} B(\alpha - k, k)$$

for $a > 0$ and $0 < k < \alpha$, where $B(a, b)$ denotes the beta function defined by

$$B(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1} dt.$$

▼**Proof**. Setting $y = 1/(1 + ax)$, we can write

$$I(k, a, \alpha) = \int_0^\infty x^{k-1}(1 + ax)^{-\alpha} dx =$$

$$= a^{-k} \int_0^1 y^{\alpha-k-1}(1 - y)^{k-1} dy$$

$$= a^{-k} B(\alpha - k, k),$$

where the final equality follows from the definition of the beta function.▲

Theorem 2.1 derives explicit expressions for the cumulative distribution functions of $M = max(X_1, X_2, \ldots, X_p)$ and $m = min(X_1, X_2, \ldots, X_p)$. Theorem 2.2 derives explicit expressions for the nth moment of $M = max(X_1, X_2, \ldots, X_p)$ and $m = min(X_1, X_2, \ldots, X_p)$. Theorem 2.3 shows how $E(M^n)$ and $E(m^n)$ vary with respect to the means, variances and the correlation coefficient in (1.2)-(1.4).

Theorem 2.1. *Let $(X_1, X_2, \ldots, X_p)$ be distributed according to (1.1). Let $M = max(X_1, X_2, \ldots, X_p)$ and $m = min(X_1, X_2, \ldots, X_p)$. Then,*

$$\Pr(M \le x) = 1 - \sum_{i=1}^p \left[1 + \frac{x}{\theta_i}\right]^{-\alpha} + \sum_{1\le i<j\le p} \left[1 + \frac{x}{\theta_i} + \frac{x}{\theta_j}\right]^{-\alpha} - \sum_{1\le i<j<k\le p} \left[1 + \frac{x}{\theta_i} + \frac{x}{\theta_j} + \frac{x}{\theta_k}\right]^{-\alpha} + \cdots + (-1)^p \left[1 + \sum_{i=1}^p \frac{x}{\theta_i}\right]^{-\alpha}$$

and

$$\Pr(m \le x) = 1 - \left[1 + \sum_{i=1}^p \frac{x}{\theta_i}\right]^{-\alpha}$$

for $x > 0$.

▼Proof. We have

$$\Pr(M \le x) = \Pr\left(\max(X_1, \ldots, X_p) \le x\right)$$

$$= \Pr\left(\bigcap_{i=1}^p (X_i \le x)\right)$$

$$= 1 - \Pr\left(\bigcap_{i=1}^p (X_i > x)\right)$$

$$= 1 - \sum_{i=1}^p \Pr(X_i > x) + \sum_{1\le i<j\le p} \Pr\left(X_i > x \bigcap X_j > x\right) - \sum_{1\le i<j<k\le p} \Pr\left(X_i > x \bigcap X_j > x \bigcap X_k > x\right) + \cdots + (-1)^p \Pr\left(\bigcap_{i=1}^p (X_i > x)\right)$$

$$= 1 - \sum_{i=1}^p \left[1 + \frac{x}{\theta_i}\right]^{-\alpha} + \sum_{1\le i<j\le p} \left[1 + \frac{x}{\theta_i} + \frac{x}{\theta_j}\right]^{-\alpha}$$

$$- \sum_{1\le i<j<k\le p} \left[1+\frac{x}{\theta_i}+\frac{x}{\theta_j}+\frac{x}{\theta_k}\right]^{-\alpha} + \cdots + (-1)^p \left[1+\sum_{i=1}^{p}\frac{x}{\theta_i}\right]^{-\alpha}$$

and

$$\begin{aligned}\Pr\ (m \le x) &= 1 - \Pr\ (m > x)\\ &= 1 - \Pr\ (\min(X_1,\ldots,X_n) > x)\\ &= 1 - \Pr\left(\bigcap_{i=1}^{p}(X_i > x)\right)\\ &= 1 - \left[1+\sum_{i=1}^{p}\frac{x}{\theta_i}\right]^{-\alpha}.\end{aligned}$$

The proof is complete.▲

Theorem 2.2. *Let $(X_1, X_2, \ldots, X_p)$ be distributed according to (1.1). Let $M = max(X_1, X_2, \ldots, X_p)$ and $m = min(X_1, X_2, \ldots, X_p)$. Let*

$$A_{i_1\cdots i_k} = \sum_{l=1}^{k}\frac{1}{\theta_{i_l}}$$

for $1 \le i_1 < \cdots < i_k \le p$. Then,

$$E(M^n) = nB(\alpha-n,n)\left[\sum_{i=1}^{p} A_i^{-n} - \sum_{1\le i<j\le p} A_{ij}^{-n} + \sum_{1\le i<j<k\le p} A_{ijk}^{-n} - \cdots - (-1)^p A_{1\cdots p}^{-n}\right]$$

and

$$E(m^n) = nB(\alpha-n,n)A_{1\cdots p}^{-n}$$

for $n < \alpha$.

▼**Proof**. Using Theorem 2.1, we can write

$$\begin{aligned}E(M^n) &= n\int_0^\infty x^{n-1}Pr(M > x)dx\\ &= n\sum_{i=1}^{p}\int_0^\infty x^{n-1}\left[1+\frac{x}{\theta_i}\right]^{-\alpha}dx - n\sum_{1\le i<j\le p}\int_0^\infty x^{n-1}\left[1+\frac{x}{\theta_i}+\frac{x}{\theta_j}\right]^{-\alpha}dx\\ &+n\sum_{1\le i<j<k\le p}\int_0^\infty x^{n-1}\left[1+\frac{x}{\theta_i}+\frac{x}{\theta_j}+\frac{x}{\theta_k}\right]^{-\alpha}dx - \cdots - n(-1)^p\int_0^\infty x^{n-1}\left[1+\sum_{i=1}^{p}\frac{x}{\theta_i}\right]^{-\alpha}dx\\ &= n\sum_{i=1}^{p}\mathrm{I}(n,A_i,\alpha) - n\sum_{1\le i<j\le p}\mathrm{I}(n,A_i,\alpha) + n\sum_{1\le i<j<k\le p}\mathrm{I}(n,A_{ijk},\alpha) - \cdots - n(-1)^p I(n,A_{1\cdots p},\alpha)\\ &= n\sum_{i=1}^{p}A_i^{-n}B(\alpha-n,n) - \sum_{1\le i<j\le p}A_{ij}^{-n}B(\alpha-n,n)\\ &+n\sum_{1\le i<j<k\le p}A_{ijk}^{-n}B(\alpha-n,n) - \cdots - n(-1)^p A_{1\cdots p}^{-1}B(\alpha-n,n),\end{aligned}$$

where the last step follows from Lemma 2.1. Also using Theorem 2.1, we can write

$$\begin{aligned}E(m^n) &= n\int_0^\infty x^{n-1}\Pr(m > x)\,dx\\ &= n\int_0^\infty x^{n-1}\left[1+\sum_{i=1}^{p}\frac{x}{\theta_i}\right]^{-\alpha}dx\\ &= nI(n,A_{1\cdots p},\alpha)\\ &= nA_{1\cdots p}^{-n}B(\alpha-n,n),\end{aligned}$$

where the last step follows from Lemma 2.1. The proof is complete.▲

Theorem 2.3. *Let $(X_1, X_2, \ldots, X_p)$ be distributed according to (1.1). Let $M = max(X_1, X_2, \ldots, X_p)$ and $m = min(X_1, X_2, \ldots, X_p)$. Then,*

(a) $\mathrm{E}(\mathrm{M}^{\mathrm{n}})$ *is an increasing function of* ρ *for* $\rho < 1/\mathrm{n}$*;*
(b) $\mathrm{E}(\mathrm{m}^{\mathrm{n}})$ *is an increasing function of* ρ *for* $\rho < 1/\mathrm{n}$*;*
(c) $\mathrm{E}(\mathrm{m}^{\mathrm{n}})$ *is an increasing function of* μ_{i}*;*

(d) $\mathrm{E}(\mathrm{m}^n)$ *is an increasing function of* σ_i*;*
(e) $\mathrm{V\,ar}(\mathrm{m})$ *is an increasing function of* ρ *for* $\rho < 1/2$*;*
(f) $\mathrm{V\,ar}(\mathrm{m})$ *is an increasing function of* μ_i*;*
(g) $\mathrm{V\,ar}(\mathrm{m})$ *is an increasing function of* σ_i.

▼**Proof.** Since

$$E(M^n) = nB(\alpha - n, n)\left[\sum_{i=1}^{p} A_i^{-n} - \sum_{1\le i<j\le p} A_{ij}^{-n} + \sum_{1\le i<j<k\le p} A_{ijk}^{-n} - \cdots - (-1)^p A_{1\cdots p}^{-n}\right]$$
$$= n\frac{\Gamma(\alpha-n)\Gamma(n)}{\Gamma(\alpha)}\left[\sum_{i=1}^{p} A_i^{-n} - \sum_{1\le i<j\le p} A_{ij}^{-n} + \sum_{1\le i<j<k\le p} A_{ijk}^{-n} - \cdots - (-1)^p A_{1\cdots p}^{-n}\right]$$
$$= \frac{n!}{(\alpha-n)\cdots(\alpha-1)}\left[\sum_{i=1}^{p} A_i^{-n} - \sum_{1\le i<j\le p} A_{ij}^{-n} + \sum_{1\le i<j<k\le p} A_{ijk}^{-n} - \cdots - (-1)^p A_{1\cdots p}^{-n}\right],$$

we see that $E(M^n)$ is a decreasing function of α for $\alpha > n$ and hence an increasing function of ρ *for* $\rho < 1/\mathrm{n}$. Since

$$E(m^n) = nB(\alpha-n,n)A_{1\cdots P}^{-N}$$
$$= NA_{1\cdots P}^{-N}\frac{\Gamma(\alpha-n)\Gamma(n)}{\Gamma(\alpha)}$$
$$= A_{1\cdots P}^{-N}\frac{n!}{(\alpha-n)\cdots(\alpha-1)},$$

we see that $E(m^n)$ is a decreasing function of α for $\alpha > n$ and hence an increasing function of ρ *for* $\rho < 1/\mathrm{n}$. Since

$$E(m^n) = nB(\alpha-n,n)A_{1\cdots P}^{-N}$$
$$= NB(\alpha-n,n)\left(\sum_{i=1}^{p}\frac{1}{\theta_i}\right)^{-n}$$
$$= nB(\alpha-n,n)(\alpha-1)^n\left(\sum_{i=1}^{p}\frac{1}{\mu_i}\right)^{-n},$$

where the last equality follows by (1.2), we see that $E(m^n)$ is an increasing function of μ_i. Since

$$E(m^n) = nB(\alpha-n,n)\left(\sum_{i=1}^{p}\frac{1}{\theta_i}\right)^{-n}$$
$$= nB(\alpha-n,n)\frac{(\alpha-1)^n(\alpha-2)^{n/2}}{\alpha^{n/2}}\left(\sum_{i=1}^{p}\frac{1}{\sigma_i}\right)^{-n},$$

where the last equality follows by (1.3), we see that $E(m^n)$ is an increasing function of σ_i. Since

$$V\,ar(m) = E(m^2) - E^2(m)$$
$$= A_{1\cdots p}^{-2}[2B(\alpha-2,2) - B^2(\alpha-1,1)]$$
$$= A_{1\cdots p}^{-2}\left[\frac{2\Gamma(\alpha-2)}{\Gamma(\alpha)} - \frac{\Gamma^2(\alpha-2)}{\Gamma^2(\alpha)}\right]$$
$$= A_{1\cdots p}^{-2}\frac{\alpha}{(\alpha-1)^2(\alpha-2)},$$

we see that $V\,ar(m)$ is a decreasing function of α for $\alpha > 2$ and hence an increasing function of ρ *for* $\rho < 1/2$. Since

$$V\,ar(m) = A_{1\cdots p}^{-2}\frac{\alpha}{(\alpha-1)^2(\alpha-2)}$$
$$= \left(\sum_{i=1}^{p}\frac{1}{\mu_i}\right)^{-2}\frac{\alpha}{\alpha-2},$$

where the last equality follows by (1.2), we see that $V\,ar(m)$ is an increasing function of μ_i. Since

$$V\,ar(m) = A_{1\cdots p}^{-2}\frac{\alpha}{(\alpha-1)^2(\alpha-2)} = \left(\sum_{i=1}^{p}\frac{1}{\sigma_i}\right)^{-2},$$

where the last equality follows by (1.3), we see that $V\,ar(m)$ is an increasing function of σ_i.▲

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