ISSN 1392 – 124X INFORMATION TECHNOLOGY AND CONTROL, 2011, Vol. 40, No. 1

EXTREMES OF A BIVARIATE PARETO DISTRIBUTION

Nooshin Hakamipour

Department of Statistics, Faculty of Mathematics and Computer Science
Amirkabir University of Technology (Tehran Polytechnic), Tehran 15914, IRAN

Adel Mohammadpour

Department of Statistics, Faculty of Mathematics and Computer Science
Amirkabir University of Technology (Tehran Polytechnic), Tehran 15914, IRAN

Saralees Nadarajah

School of Mathematics, University of Manchester
Manchester M13 9PL, UK
e-mail: mbbsssn2@manchester.ac.uk

crossref http://dx.doi.org/10.5755/j01.itc.40.1.195

Abstract. Distributional properties of min(*X, Y*) and max(*X, Y*) are studied when (*X, Y*) has a bivariate Pareto distribution. Extensions are given to the multivariate case.

Keywords: Bivariate Pareto distributions, extremes, moments.

1. Introduction

Bivariate Pareto distributions are popular models in many applied areas. They are very versatile and a variety of uncertainties can be usefully modeled by them. We mention: modeling of radiation carcinogenesis (Rachev *et al.* [1]), performance measures for general systems (Nadarajah and Kotz [2]), reliability (Hanagal [3]; Navarro *et al.* [4]), modeling of drought (Nadarajah [5]), modeling of dependent heavy tailed risks with a non-zero probability of simultaneous loss (Asimit *et al.* [6]), and modeling of daily exchange rate data (Papadakis and Tsionas [7]).

Let (X, Y) be a bivariate Pareto random vector. In the mentioned applications, X and Y could be the lifetimes of two components, the magnitudes of stress and strength components, drought intensities for two regions, risks for two insurance events, exchange rates in two time periods, and so on. So, it is important to know which of the two variables, X and Y, is larger or smaller.

Let $S = \min(X, Y)$ and $T = \max(X, Y)$. The aim of this note is to study the distributions of S and T when (X, Y) has a bivariate Pareto distribution. Studies of this kind have been considered by several authors. Ker [8] studies the distribution T when (X, Y) has a bivariate normal distribution. Lien [9] studies the distributions of S and T when (X, Y) has a bivariate lognormal distribution. Aksomaitis and Burauskaite-Harju [10] study the distribution of $\max(X_1, X_2, \ldots, X_n)$ and its moments when $(X_1, X_2, \ldots, X_n)$ has a multivariate normal distribution.

It seems, however, that the distributions of S and T have not been studied when (X, Y) has a bivariate Pareto distribution. This note provides the first such study. We take (X, Y) to have the simplest bivariate Pareto distribution due to Muliere and Scarsini [11]: the one given by the following joint survival function:

$$\overline{F}_{X,Y}(x, y) = \left(\frac{x}{\beta}\right)^{-\lambda_1} \left(\frac{y}{\beta}\right)^{-\lambda_2} \times \left\{\max\left(\frac{x}{\beta}, \frac{y}{\beta}\right)\right\}^{-\lambda_0} \quad (1.1)$$

for $\lambda_i > 0$, $i = 0, 1, 2$, where $0 < \beta \leq \min\,(x, y) < \infty$. The corresponding joint cumulative distribution function is

$$F_{X,Y}(x,y) = 1-\left(\frac{x}{\beta}\right)^{-\lambda_0-\lambda_1}-\left(\frac{y}{\beta}\right)^{-\lambda_0-\lambda_2} +\left(\frac{x}{\beta}\right)^{-\lambda_1}\left(\frac{y}{\beta}\right)^{-\lambda_2} \times\left\{\max\left(\frac{x}{\beta},\frac{y}{\beta}\right)\right\}^{-\lambda_0}.$$

The cumulative distribution functions of S and T are

$$F_S(s) = 1-\left(\frac{s}{\beta}\right)^{-(\lambda_0+\lambda_1+\lambda_2)}$$

and

$$F_T(t) = 1-\left(\frac{t}{\beta}\right)^{-\lambda_0-\lambda_1}+\left(\frac{t}{\beta}\right)^{-\lambda_0-\lambda_1-\lambda_2} -\left(\frac{t}{\beta}\right)^{-\lambda_0-\lambda_2}.$$

The corresponding probability density functions are:

$$f_S(s) = (\lambda_0+\lambda_1+\lambda_2)\,\beta^{\lambda_0+\lambda_1+\lambda_2}s^{-\lambda_0-\lambda_1-\lambda_2-1},$$

and

$$f_T(t) = (\lambda_0+\lambda_1)\,\beta^{\lambda_0+\lambda_1}t^{-\lambda_0-\lambda_1-1} -(\lambda_0+\lambda_1+\lambda_2)\,\beta^{\lambda_0+\lambda_1+\lambda_2} \times t^{-\lambda_0-\lambda_1-\lambda_2-1} +(\lambda_0-\lambda_2)\,\beta^{\lambda_0+\lambda_2}t^{-\lambda_0-\lambda_2-1}, \quad (1.2)$$

where $0<\beta<t<s<\infty$ and $\lambda_i>0$, $i=0,1,2$.

In Section 2, we provide expressions for $E(S)$, $Var(S)$, $E(T)$, $Var(T)$, and examine the effects of λ_i on them. Section 3 provides an extension to a multivariate case.

2. Main result

Theorem 1 is our main result.

Theorem 1. *Let (X,Y) be distributed according to (1.1). Let $S=\min(X,Y)$, $T=\max(X,Y)$ and $\lambda=\lambda_0+\lambda_1+\lambda_2$. Then*

$$E(S) = \frac{\lambda\beta}{\lambda-1},\ \lambda>1,$$

$$E\left(S^2\right) = \frac{\lambda\beta^2}{\lambda-2},\ \lambda>2,$$

$$Var(S) = \frac{\beta^2\lambda}{(\lambda-2)(\lambda-1)^2},\ \lambda>2,$$

$$E(T) = \frac{\beta(\lambda_0+\lambda_1)}{\lambda_0+\lambda_1-1}+\frac{\beta\lambda}{1-\lambda} +\frac{\beta(\lambda_0+\lambda_2)}{\lambda_0+\lambda_2-1}, \quad \lambda_0>\max\{0,1-\lambda_1,1-\lambda_2\},$$

$$E\left(T^2\right) = \frac{\beta^2(\lambda_0+\lambda_1)}{\lambda_0+\lambda_1-2}+\frac{\beta^2\lambda}{2-\lambda} +\frac{\beta^2(\lambda_0+\lambda_2)}{\lambda_0+\lambda_2-2}, \quad \lambda_0>\max\{0,2-\lambda_1,2-\lambda_2\},$$

and

$$Var(T) = \beta^2\left[\frac{\lambda_0+\lambda_1}{\lambda_0+\lambda_1-2}+\frac{\lambda_0+\lambda_2}{\lambda_0+\lambda_2-2} +\frac{\lambda}{2-\lambda} -\left(\frac{\lambda_0+\lambda_1}{\lambda_0+\lambda_1-1}+\frac{\lambda_0+\lambda_2}{\lambda_0+\lambda_2-1} +\frac{\lambda}{1-\lambda}\right)^2\right], \quad \lambda_0>\max\{0,2-\lambda_1,2-\lambda_2\}.$$

Furthermore, we have the following statements holding:

(I) $E(S)$ is monotonically increasing with respect to β for $\lambda>1$,

(II) $E(S)$ is monotonically decreasing with respect to λ and λ_i, $i=0,1,2$ for $\lambda>1$,

(III) $Var(S)$ is monotonically increasing with respect to β for $\lambda>2$,

(IV) $Var(S)$ is monotonically decreasing with respect to λ and λ_i for $i=0,1,2$ for $\lambda>2$,

(V) $E(T)$ is monotonically increasing with respect to β for $\lambda_0>\max\{1-\lambda_1,1-\lambda_2,0\}$,

(VI) $E(T)$ is monotonically decreasing with respect to λ_i for $i=0,1,2$ for $\lambda_0>\max\{1-\lambda_1,1-\lambda_2,0\}$,

(VII) $Var(T)$ is monotonically increasing with respect to β,

$(VIII)$ $Var(T)$ is monotonically decreasing with respect to λ_i for $i=0,1,2$.

Proof: The given expressions for $E(S)$, $E(S^2)$ and $Var(S)$ follow by using:

$$E(S) = \int_{\beta}^{\infty} \lambda \beta^{\lambda} s^{-\lambda} ds,$$

$$E\left(S^2\right) = \int_{\beta}^{\infty} \lambda \beta^{\lambda} s^{1-\lambda} ds$$

and $Var(S) = E(S^2) - [E(S)]^2$. Since, using (1.2),

$$E(T) = \int_{\beta}^{\infty} t f_T(t) dt = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{\beta}^{\infty} (\lambda_0 + \lambda_1) \beta^{\lambda_0+\lambda_1} t^{-\lambda_0-\lambda_1} dt \\ &= \frac{\beta\left(\lambda_0 + \lambda_1\right)}{\lambda_0 + \lambda_1 - 1}, \ \lambda_0 + \lambda_1 > 1, \\ I_2 &= \int_{\beta}^{\infty} -\lambda \beta^{\lambda} t^{-\lambda} dt \\ &= \frac{\beta \lambda}{1 - \lambda}, \ \lambda > 1, \\ I_3 &= \int_{\beta}^{\infty} \left(\lambda_0 + \lambda_2\right) \beta^{\lambda_0+\lambda_2} t^{-\lambda_0-\lambda_2} dt \\ &= \frac{\beta\left(\lambda_0 + \lambda_2\right)}{\lambda_0 + \lambda_2 - 1}, \ \lambda_0 + \lambda_2 > 1, \end{aligned}$$

we obtain the given expression for $E(T)$. Similarly, using

$$E\left(T^2\right) = \int_{\beta}^{\infty} t^2 f_T(t) dt = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{\beta}^{\infty} \left(\lambda_0 + \lambda_1\right) \beta^{\lambda_0+\lambda_1} t^{-\lambda_0-\lambda_1+1} dt \\ &= \frac{\beta^2\left(\lambda_0 + \lambda_1\right)}{\lambda_0 + \lambda_1 - 2}, \ \lambda_0 + \lambda_1 > 2, \\ I_2 &= \int_{\beta}^{\infty} -\lambda \beta^{\lambda} t^{-\lambda+1} dt \\ &= \frac{\beta^2 \lambda}{2 - \lambda}, \ \lambda > 2, \\ I_3 &= \int_{\beta}^{\infty} \left(\lambda_0 + \lambda_2\right) \beta^{\lambda_0+\lambda_2} t^{-\lambda_0-\lambda_2+1} dt \\ &= \frac{\beta^2\left(\lambda_0 + \lambda_2\right)}{\lambda_0 + \lambda_2 - 2}, \ \lambda_0 + \lambda_2 > 2, \end{aligned}$$

and $Var(T) = E(T^2) - [E(T)]^2$, we obtain the given expression for $Var(T)$.

The remainder of the theorem is proved as follows:

(I) We have

$$\frac{\partial E(S)}{\partial \beta} = \frac{\lambda}{\lambda - 1} > 0, \ \lambda > 1,$$

so statement (I) follows.

(II) We have

$$\frac{\partial E(S)}{\partial \lambda_i} = \frac{-\beta}{\left(\lambda - 1\right)^2} < 0, \ \lambda > 1,$$

and

$$\frac{\partial E(S)}{\partial \lambda} = \frac{-\beta}{(\lambda - 1)^2} < 0, \ \lambda > 1,$$

so statement (II) follows.

(III) We have

$$\frac{\partial Var(S)}{\partial \beta} = \frac{2\beta\lambda}{\left(\lambda - 2\right)\left(\lambda - 1\right)^2}.$$

Since $\beta > 0$ and $\lambda > 2$, $\partial Var(S)/\partial\beta > 0$, so statement (III) follows.

(IV) We have

$$\frac{\partial Var(S)}{\partial \lambda_i} = -2\beta^2 \frac{\left(\lambda^2 - 1 - \lambda\right)}{\left(\lambda - 1\right)^3 \left(\lambda - 2\right)^2}, \ i = 0, 1, 2,$$

$$Var(S) = \frac{\lambda \beta^2}{(\lambda - 2)(\lambda - 1)^2}, \ \lambda > 2$$

and

$$\frac{\partial Var(S)}{\partial \lambda} = -2\beta^2 \left(\frac{\lambda^2 - \lambda - 1}{(\lambda - 1)^3 (\lambda - 2)^2} \right).$$

Since $\lambda > 2$ and $(\lambda^2 - \lambda - 1) > 0$, we have

$$\frac{\partial Var(S)}{\partial \lambda_i} < 0,$$

and

$$\frac{\partial Var(S)}{\partial \lambda} < 0,$$

so statement (IV) follows.

(V) We have

$$\begin{aligned} \frac{\partial E(T)}{\partial \beta} &= \frac{\lambda_0 + \lambda_1}{\lambda_0 + \lambda_1 - 1} + \frac{\lambda_0 + \lambda_2}{\lambda_0 + \lambda_2 - 1} - \frac{\lambda}{\lambda - 1}, \\ &\lambda_0 > \max\left\{1 - \lambda_1, 1 - \lambda_2, 0\right\}. \end{aligned}$$

Since $(\lambda_0 + \lambda_1)/(\lambda_0 + \lambda_1 - 1) > \lambda/(\lambda - 1)$, $\partial E(T)/\partial\beta > 0$, so statement (V) follows.

(VI) We have

$$\frac{\partial E(T)}{\partial \lambda_0} = \beta\left[\frac{1}{\lambda_0+\lambda_1-1} + \frac{1}{\lambda_0+\lambda_2-1} - \frac{1}{\lambda-1} - \left(\frac{\lambda_0+\lambda_1}{(\lambda_0+\lambda_1-1)^2} + \frac{\lambda_0+\lambda_2}{(\lambda_0+\lambda_2-1)^2} - \frac{\lambda}{(\lambda-1)^2}\right)\right],$$

$$\frac{\partial E(T)}{\partial \lambda_1} = \frac{-\beta\lambda_2\left(-2+2\lambda_0+2\lambda_1+\lambda_2\right)}{(\lambda_0+\lambda_1-1)^2(\lambda-1)^2}$$

and

$$\frac{\partial E(T)}{\partial \lambda_2} = \frac{-\beta\lambda_1(-2+2\lambda_0+\lambda_1+2\lambda_2)}{(\lambda_0+\lambda_2-1)^2(\lambda-1)^2}.$$

Since $1/(\lambda_0+\lambda_2-1) < (\lambda_0+\lambda_2)/(\lambda_0+\lambda_2-1)^2$ and $1/(\lambda_0+\lambda_1-1)-1/(\lambda-1) < (\lambda_0+\lambda_1)/((\lambda_0+\lambda_1-1)^2)-\lambda/(\lambda-1)^2$, $\partial E(T)/\partial\lambda_0 < 0$. Since $\lambda_0+\lambda_1 > 1$ and $\lambda_2 > 0$, we have $(-2+2\lambda_0+2\lambda_1+\lambda_2) > 0$, so $\partial E(T)/\partial\lambda_1 < 0$. Since $\lambda_0+\lambda_2 > 1$ and $\lambda_1 > 0$, so $\partial E(T)/\partial\lambda_2 < 0$.

The calculations required for (VII) and $(VIII)$ are as routine as those for (V) and (VI), but they are lot more lengthy. The proof is complete. □

3. Multivariate extension

Consider the following multivariate generalization of (1.1):

$$\overline{F}_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = \left(\frac{x_1}{\beta}\right)^{-\lambda_1}\left(\frac{x_2}{\beta}\right)^{-\lambda_2} \cdots \left(\frac{x_n}{\beta}\right)^{-\lambda_n}\left\{\max\left(\frac{x_1}{\beta},\frac{x_2}{\beta},\ldots,\frac{x_n}{\beta}\right)\right\}^{-\lambda_0},$$

where $0 < \beta \leq \min\,(x_1,x_2,\ldots,x_n) < \infty$, and $\lambda_i > 0$, $i = 0,1,\ldots,n$ (Kotz *et al.* [12, page 595]). Theorem 2 provides the multivariate analogue of Theorem 1 for $\min(X_1,X_2,\ldots,X_n)$.

Theorem 2. *Let $S = \min(X_1,X_2,\ldots,X_n)$ and $\lambda = \sum_{i=0}^{n}\lambda_i$. Then*

$$F_S(s) = 1 - \left(\frac{s}{\beta}\right)^{-\lambda},$$

$$f_S(s) = \lambda s^{-\lambda-1}\beta^{\lambda},$$

$$E(S) = \frac{\lambda\beta}{\lambda-1},\ \lambda > 1,$$

and

$$Var(S) = \frac{\beta^2\lambda}{(\lambda-2)(\lambda-1)^2},\ \lambda > 2.$$

Furthermore, we have the following statements holding:

(I) *$E(S)$ is a monotonically increasing function of β for $\lambda > 1$,*

(II) *$E(S)$ is a monotonically decreasing function of λ_i,*

(III) *$Var(S)$ is a monotonically increasing function of β for $\lambda > 2$,*

(IV) *$Var(S)$ is a monotonically decreasing function of λ for $\lambda > 2$,*

(V) *$Var(S)$ is a monotonically decreasing function of λ_i.*

Proof: The given expressions for $F_S(s)$, $f_S(s)$, $E(S)$ and $Var(S)$ are obvious. Since

$$\frac{\partial E(S)}{\partial\beta} = \frac{\lambda}{\lambda-1} > 0,\ \lambda > 1,$$

statement (I) follows. Since

$$\frac{\partial E(S)}{\partial\lambda_i} = \frac{-\beta}{(\lambda-1)^2} < 0,\ i = 0,1,\ldots,n,$$

statement (II) follows. Since

$$\frac{\partial Var(S)}{\partial\beta} = \frac{2\lambda\beta}{(\lambda-2)(\lambda-1)^2} > 0,\ \lambda > 2,$$

statement (III) follows. Since

$$\frac{\partial Var(S)}{\partial\lambda} = -2\beta^2\left(\frac{\lambda^2-\lambda-1}{(\lambda-2)^2(\lambda-1)^3}\right) < 0,\ \lambda > 2,$$

statement (IV) follows. A similar analysis shows statement (V). □

References

[1] **S. T. Rachev, C. F. Wu, A. Y. Yakovlev.** A bivariate limiting distribution of tumor latency time. *Mathematical Biosciences*, 1995, 127, 127-147.

[2] **S. Nadarajah, S. Kotz.** Performance measures for some bivariate Pareto distributions. *International Journal of General Systems*, 2006, 35, 387-393.

[3] **D. D. Hanagal.** Note on estimation of reliability under bivariate Pareto stress-strength model. *Statistical Papers*, 1997, 38, 453-459.

[4] **J. Navarro, J. M. Ruiz, C. J. Sandoval.** Properties of systems with two exchangeable Pareto components. *Statistical Papers*, 2008, 49, 177-190.

[5] **S. Nadarajah.** A bivariate Pareto model for drought. *Stochastic Environmental Research and Risk Assessment*, 2009, 23, 811-822.

[6] **A. V. Asimit, E. Furman, R. Vernic.** On a multivariate Pareto distribution. *Insurnace: Mathematics and Economics*, 2010, 46, 308-316.

[7] **E. N. Papadakis, E. G. Tsionas.** Multivariate Pareto distributions: inference and financial applications. *Communications in Statistics—Theory and Methods*, 2010, 39, 1013-1025.

[8] **A. P. Ker.** On the maximum of bivariate normal random variables. *Extremes*, 2005, 4, 185-190.

[9] **D. Lien.** On the minimum and maximum of bivariate lognormal random variables. *Extremes*, 2005, 8, 79-83.

[10] **A. Aksomaitis, A. Burauskaite-Harju.** The moments of the maximum of normally distributed dependent values. *Information Technology and Control*, 2009, 38, 301-302.

[11] **P. Muliere, M. Scarsini.** Characterization of a Marshall-Olkin type class of distributions. *Annals of the Institute of Statistical Mathematics*, 1987, 39, 429-441.

[12] **S. Kotz, N. Balakrishnan, N. L. Johnson.** Continuous Multivariate Distributions: Models and Applications, second edition, volume 1. *John Wiley and Sons, New York*, 2000.

Received October 2009.