ITC 2/46
Journal of Information Technology and Control
Vol. 46 / No. 2 / 2017
pp. 194-204
DOI 10.5755/j01.itc.46.2.14038


The Probability of Stability Estimation of an Arbitrary Order DPCM Prediction Filter: Comparison Between the Classical Approach and the Monte Carlo Method

Received 2016/01/26 | Accepted after revision 2017/04/12

http://dx.doi.org/10.5755/j01.itc.46.2.14038

# The Probability of Stability Estimation of an Arbitrary Order DPCM Prediction Filter: Comparison Between the Classical Approach and the Monte Carlo Method

**Nikola B. Danković, Dragan S. Antić, Saša S. Nikolić, Staniša Lj. Perić**

University of Niš, Faculty of Electronic Engineering, Department of Control Systems, Aleksandra Medvedeva 14, Niš 18000, Republic of Serbia, e-mail: {nikola.dankovic, dragan.antic, sasa.s.nikolic, stanisa.peric}@elfak.ni.ac.rs

**Zoran H. Perić**

University of Niš, Faculty of Electronic Engineering, Department of Telecommunications, Aleksandra Medvedeva 14, Niš 18000, Republic of Serbia, e-mail: {zoran.peric}@elfak.ni.ac.rs

**Aleksandar V. Jocić**

University of Niš, Faculty of Electronic Engineering, Department of Measurements, Aleksandra Medvedeva 14, Niš 18000, Republic of Serbia, e-mail: {aleksandar.jocic}@elfak.ni.ac.rs

**Corresponding author:** nikola.dankovic@elfak.ni.ac.rs

This paper presents the stability analysis of the linear recursive (prediction) filters with higher-order predictors in a DPCM (differential pulse-code modulation) system, where traditional methods become too difficult and complex. Stability conditions for the third- and fourth-order predictor are given by using the Schur–Cohn stability criterion. The probability of stability estimation is performed by using the Monte Carlo method. Verification of the proposed method is performed for lower-order predictors (the first- and second-order). We calculated numerical values of the probability of stability for higher-order predictors and previously experimentally obtained parameters. With large enough number of trials (samples) in Monte Carlo simulation, we reach the desired accuracy.

**KEYWORDS:** linear prediction, normal distribution, predictor coefficients, stability conditions, Monte Carlo integration.

## Introduction

For half a century, differential pulse code modulation (DPCM) has been one of the most effective techniques for signal processing and transmission based on the prediction filter. This technique is widely used in telecommunications, speech [16, 22] and image coding [20, 28], medical research [10, 15, 21], etc.

A prediction (recursive) filter is a central part and the basis of each DPCM system. It is located in a negative feedback loop. Because of the negative feedback, although basically a telecommunication system, DPCM is also suitable for the analysis in control systems theory. Some traditional stability analyses of DPCM transmission system have been already performed for the first [19], second [24], and higher-order predictors [25]. The stability of the prediction filter (the linear part of DPCM system), is a sufficient condition for stability of the whole system.

Generally, linear prediction is commonly used in various areas such as adaptive filtering, system identification, spectral estimation, and speech [4]. Linear prediction, where the prediction of the current sample is calculated as the linear combination of the previous samples, is the basis of the DPCM system. The prediction gain significantly increases up to four-order predictors when it gets into saturation [16]. On the other hand, a sensitivity analysis for the prediction filter is performed in [9].

In this paper, we discuss stochastic stability of the prediction filter. In practice, each real system is imperfect in some way [1, 2, 3, 17]. It means that system parameters are stochastic variables, not deterministic. In this case, predictor coefficients have normal distribution around nominal value. Previously performed classical methods for stability analysis are not applicable for the systems with stochastic parameters, because, in this case, we actually estimate only which probability allows the system to be stable. For these reasons, in previous papers we introduced the term “probability of stability” [1, 2, 17]. The great importance of the presented method is in its application in practice. The selection of the appropriate parameter values, for which the system has the largest probability of stability, provides the stability of the system and correctness of its work.

We have already used the method of stability estimation for the first- and second-order predictor [8]. For the higher-order predictors, it becomes more and more difficult to use the classical integration method for obtaining desired probability of stability. Despite some attempts, only theoretical approach of stability analysis for higher-order prediction filters has been given till now without adequate numerical results [5, 23]. In this paper we use the Monte Carlo method [11, 12] for numerical integration.

First, we use the Monte Carlo method to verify the results obtained by the classical method for the probability of stability estimation. We perform experiments for the first- and second-order predictors and compare the obtained results with the results already obtained by using classical integration. Then, we apply the Monte Carlo method to the third- and the fourth-order systems and determined numerical values for the probability of stability. In addition, we make a comparison with some alternative approximate methods for probability of stability estimation with regard to the error and distinctness of the methods.

## Theoretical Background of a Prediction Filter in the DPCM Transmission System

The DPCM system is suitable for digitalization and transmission of highly correlated signals. This quality of the system is provided by a prediction filter in the negative feedback loop. The prediction filter estimates the actual sample value based on one or more previous samples of input (source) signal. A number of previous samples, which are used for prediction, determines predictor order $k$.

Differential pulse code modulation system is shown in Fig. 1. A DPCM encoder (Fig. 1a) consists of the quantizer, inverse quantizer, and a predictor. Prediction filters in the encoder and decoder are marked with dashes.

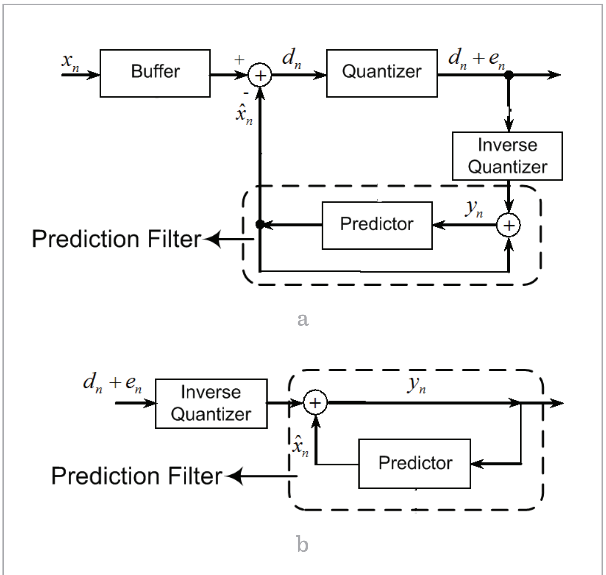
The difference between input sample $x_n$ and its predicted (estimated) value $\hat{x}_n$ is led to quantizer input:

$$d_n = x_n - \hat{x}_n \,. \tag{1}$$

If estimation of $\hat{x}_n$ is correct, samples of difference $d_n$ have significantly less amplitude dynamics in comparison with $x_n$. In this way, quantization of a difference signal can be performed by a smaller number of amplitude levels which provides bit-rate saving, i.e., the prediction gain [16].

**Figure 1**

Block scheme of a DPCM system, a) Encoder b) Decoder




The difference (1) is quantized and a quantization error $e_n$ occurs due to quantization of the difference $d_n$. Finally, the quantized difference is encoded and the digital value is formed. Reconstructed sample $y_n$ is actually a source signal sample with the quantization error $e_n$ added:

$$y_n = d_n + e_n + \hat{x}_n = x_n + e_n . \quad (2)$$

For the linear predictor, the predicted (estimated) value $\hat{x}_n$ is calculated as a linear combination of the previously quantized reconstructed samples $y_{n-i}$:

$$\hat{x}_n = \sum_{i=1}^{k} a_i y_{n-i} , \quad (3)$$

where $a_i$, $i = 1,2,...,k$ are predictor coefficients.

From the equations above we can see that estimation accuracy of input signal samples value (3) directly depends on quality selection of predictor coefficients values. This accuracy has a further influence on the quantization error and the prediction gain. Low-quality selection of predictor coefficients could cause bigger difference signal $d_n$ than input signal $x_n$, and also a bigger quantization error. Afterwards, multiplication of total error through feedback loop may appear, which finally leads to system failure. Therefore, DPCM stability mostly depends on correct selection of linear predictor coefficients.

According to all the remarks above, linear prediction filters in the encoder (Fig. 1a) and decoder (Fig. 1b), as the main parts of the whole system, are of special interest for the stability estimation.

Relation (3) describes the $k$-th order linear predictor and it can be rewritten in $z$-domain:

$$\hat{X}(z) = \left( \sum_{i=1}^{k} a_i z^{-i} \right) Y(z) . \quad (4)$$

The transfer function of the predictor is:

$$W_P(z) = \frac{\hat{X}(z)}{Y(z)} = \sum_{i=1}^{k} a_i z^{-i} . \quad (5)$$

The transfer function of the prediction filter in the encoder has the following form:

$$W_R(z) = \frac{W_P(z)}{1 - W_P(z)} = \left( \sum_{i=1}^{k} a_i z^{-i} \right) \left( 1 - \sum_{i=1}^{k} a_i z^{-i} \right)^{-1} . \quad (6)$$

The transfer function of the prediction filter in the decoder is:

$$W_R^D(z) = \frac{1}{1 - W_P(z)} = \left( 1 - \sum_{i=1}^{k} a_i z^{-i} \right)^{-1} . \quad (7)$$

## Stability Analysis of the Linear Prediction Filter

The main goal of prediction filter analysis is to determine the predictor coefficients for which the system has the best performance. Our task is to examine the prediction filter stability depending on different values of predictor coefficients.

The stability of the prediction filter is sufficient for the stability of the whole system, so prediction filter stability consideration is very important during the system design process. The basic requirement is that predictor coefficients are located inside the stability region in parametric space or very close to this region.

That means that we need to determine predictor parameter values for which the prediction filter is stable or very close to the stability domain.

Prediction filters in the encoder and decoder are stable if all the poles of the transfer functions (6) and (7) lie inside the unit circle, i.e., if the characteristic equation (which is the same for the both filters):

$$1-\sum_{i=1}^{k} a_i z^{-i} = 0 \, , \tag{8}$$

has all its zeroes inside the unit circle. The equation (8) can be written as:

$$z^k - \sum_{i=1}^{k} a_i z^{k-i} = 0 \, . \tag{9}$$

Stability conditions of the system described with characteristic equation (9) can be determined with several stability criteria. Thus, in [17] we used the Routh–Hurwitz stability criterion.

Predictor coefficients are not deterministic in practice. That is why a DPCM system is not perfect as any other real system, too. Sometimes, these imperfections do not have any visible effect on the system performances, but in many cases this effect cannot be neglected. Some system properties, such as stability or dynamical response, are directly dependent on this. Mathematically, the imperfections are variations of system coefficients around the nominal values of the coefficients [1, 2, 17].

In this paper, we consider a type of a real system which is stable with a certain probability. That is the reason why we need to introduce the term probability of stability, instead of traditional stability of the system. The probability of stability is defined as [17]:

$$P = \int \cdots \int_{S_k} f\left(a_1, a_2, \ldots, a_k\right) da_1 da_2 \cdots da_k \, , \tag{10}$$

where $S_k$ is the stability region and $f\left(a_1, a_2, \ldots, a_k\right) = \prod_{i=1}^{k} f_i\left(a_i\right)$ is the total density function.

Based on these facts, we can estimate the stability of an arbitrary order prediction filter, but some difficulties can occur during calculations. Namely, the shapes of the stability regions become too complex and not convex for $k \geq 3$ [5, 7, 26]. Some geometric properties of the stability region $S_k$ are given in [23]. The shape of $S_k$ was determined for low-order prediction filters, and only general properties were given for higher-order filters. Keeping in mind that we perform stochastic stability [13], not deterministic, it is more difficult to obtain the desired numerical results. This is the reason why we propose a new improved method for the probability of stability calculation by using the Monte Carlo integration.

However, we will first verify the accuracy of the Monte Carlo method for already performed stability estimation for the first- and second-order predictors.

## The Monte Carlo Method

The Monte Carlo method provides approximate numerical solutions to various problems by performing statistical sampling experiments on a computer. The method is especially useful for mathematical problems which are too complicated to solve analytically [11, 12, 17].

In this paper, we use the Monte Carlo method for numerical estimation of multidimensional definite integrals which are very difficult to solve by classical integration, especially integrals in higher dimensions. In some cases, bounds of integration are too complex for determination, and some approximations have to be introduced. By using the proposed Monte Carlo numerical integration, we can obtain numerical values of definite integrals of arbitrary dimension with desired accuracy. Let us notice that we have probability density of parameters (in this case predictor coefficients).

In this paper, we firstly use the Monte Carlo method to confirm the results obtained by the classical method for the probability of stability estimation using (10). These verification experiments were performed for the first- and second-order predictor (normal distribution of predictor coefficients). Later, we use the proposed method for estimation of probability integrals for the third- and fourth-order, where classical integration is too complex. The random number generator is used to generate the values of the parameters with normal distribution. The experiments are performed over different numbers of samples (trials). The testing of the probability of stability using the Monte Carlo method is much easier, because there is no need for integration over the region of stability [6, 7], but only the limits of the stability region are required. The probability of stability is calculated as the quotient of the number of favourable samples (samples that belong to the region of stability) and the total number of samples.

## The probability of Stability Estimation by Using Clasical Integration and the Monte Carlo Method in the case of the First- and Second-Order Predictors

In the case of the first-order predictor, the stability region is well-known [8] and presented with the following condition:

$$-1 \le a_1 \le 1 . \tag{11}$$

Let us denote the stability region described by (11) with $S_1$. The probability of stability is given by:

$$P_{S_1} = \frac{1}{\sigma_1 \sqrt{2\pi}} \int_{-1}^{1} \exp\left[ -\frac{1}{2}\left( \frac{a_1 - \overline{a}_1}{\sigma_1} \right)^2 \right] da_1 , \tag{12}$$

where $\overline{a}_1$ is the mean value of $a_1$, and $\sigma_1$ is the standard deviation of parameter $a_1$.

Now, we will test the Monte Carlo method for the probability of stability for the previously calculated values using classical integration.

For $\overline{a}_1 = 0.81$ and $\sigma_1 = 0.2$, the obtained value for the probability of stability using classical integration (12) is 0.8289 (82.89%) [8]. In this paper, we apply the Monte Carlo method to estimate the probability of stability for the same distribution parameters of the predictor coefficient. The random number generator was used to generate the values of the predictor coefficients with normal distribution.

Verification experiment is performed with 10,000 trials. We counted the number of parameter values which satisfy (11) and obtained the probability value of 0.8331 (83.31%). As we can see, the deviation from the value obtained by the classical method is less than 0.005. If we need better accuracy, we can get higher number of trials. For repeated experiment with 100000 trials, we obtained the probability of stability value of 0.8279 (82.79%). In order to obtain the probability of stability with desired accuracy, we performed two more experiments. With 1,000,000 trials we obtain the value of 0.8281 (82.81%). It means that we still did not reach the accuracy to three decimal places. Finally, after we repeat the experiment, this time with 10,000,000 trials, we obtain the value of 0.8290 (82.90%). Thus we reached the desired accuracy.

The accuracy of the Monte Carlo Method depends on the number of trials ($\sim \frac{1}{\sqrt{N}}$), but also on the standard deviation [14]:

$$e \approx \frac{\sigma}{\sqrt{N}} . \tag{13}$$

Detailed error analysis is not necessary for the purpose of probability stability estimation of the DPCM prediction filter. We can easily achieve satisfied accuracy, but we perform more experiments again for the second-order predictor, because we also know the exact value of the probability of stability, and furthermore, for higher-order predictors we adopt a number of trials large enough for our purpose. In order to compare accuracy of different methods, we will show how the errors occur if we use some alternative approximate methods [6, 7, 27].

For the second-order predictor, the stability region $S_2$, in the parametric space, $a_1, a_2$, is given with the following conditions:

$$1 + a_1 - a_2 \ge 0, \quad 1 - a_1 - a_2 \ge 0, \quad a_2 \ge -1. \tag{14}$$

The probability density function (PDF) for normal distribution has the following form [1, 2, 8, 17]:

$$f_2(a_1, a_2) = \frac{1}{\sigma_1 \sqrt{2\pi}} \frac{1}{\sigma_2 \sqrt{2\pi}} \times \\ \times \exp\left[ -\frac{1}{2}\left( \frac{a_1 - \overline{a}_1}{\sigma_1} \right)^2 - \frac{1}{2}\left( \frac{a_2 - \overline{a}_2}{\sigma_2} \right)^2 \right], \tag{15}$$

where $\sigma_1$ and $\sigma_2$ are standard deviations, while $\overline{a}_1$ and $\overline{a}_2$ are the mean values of predictor coefficients $a_1$ and $a_2$, respectively.

The probability of stability is derived from:

$$P_{S_2} = \frac{\iint\limits_{S_2} f_2\left(a_1, a_2\right) da_1 da_2}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f_2(a_1, a_2) da_1 da_2}. \tag{16}$$

**Remark 1.** The integral in the denominator in (16) presents total probability and is equal to 1.

For the obtained values of predictor coefficients $a_1$ and $a_2$, we calculated the mean value and the standard deviation for different values of frame length, $M$. In the case when $M = 50$ and the following mean (nominal) values of predictor coefficients: $\overline{a}_1 = 1.292$, $\overline{a}_2 = -0.410$, and standard deviations: $\sigma_1 = 0.206$, $\sigma_2 = 0.205$, respectively, we obtained the value of 0.6556 (65.56%) for probability of stability of prediction filter, by calculating integral given by (16).

Now, we test the Monte Carlo method for the second-order predictor. We performed four experiments, also (with 10000, 100000, 1000000 and 10000000 trials). Obtained values for probability of stability are given in Table 1. As we can see, after $10^7$ trials, an estimation error is less than 0.001.

**Table 1**

Probability of stability using classical integration and the Monte Carlo method for mean values $\overline{a}_1 = 1.292$, $\overline{a}_2 = -0.410$, and standard deviations $\sigma_1 = 0.206$, $\sigma_2 = 0.205$ of the second-order predictor coefficients

| Methods | Classical | Monte Carlo | | | |
|---|---|---|---|---|---|
| Trials | / | $10^4$ | $10^5$ | $10^6$ | $10^7$ |
| Probability of stability | 0.6556 | 0.6542 | 0.6545 | 0.6559 | 0.6558 |
| Error | / | 0.0014 | 0.0011 | 0.0003 | 0.0002 |

According to the previous estimation error analysis, for the higher-order predictor, we perform experiments with $10^7$ trials (the accuracy approximately to three decimal places).

As we have already concluded, for higher-order systems, integration becomes too complicated and the Monte Carlo method, which we tested in this section, can be very good and reliable alternative for the probability stability estimation.

## Application of the Monte Carlo Method for Stability Estimation of the Prediction Filter with Higher-Order Predictors

Generally, for the $k$-th order prediction filter ($k \geq 3$), the probability of stability could be determined using the relations (10), too. However, the calculation is too complex. The limits of the stability region $S_k$ are usually complex mathematical expressions and it is difficult to determine the probability of stability because it is necessary to perform integration over the region of stability [7, 26]. That is the reason why we estimate the probability of stability later on.

We will use the Schur–Cohn stability criterion [18]. We adapted it to the form of characteristic equation (9) and built appropriate determinants:

$$\Delta_i = \left|\begin{array}{cccc|cccc}
-a_k & 0 & \cdots & 0 & 1 & -a_1 & \cdots & -a_{i-1} \\
-a_{k-1} & -a_k & \cdots & 0 & 0 & 1 & \cdots & -a_{i-2} \\
-a_{k-2} & -a_{k-1} & \cdots & 0 & 0 & 0 & \cdots & -a_{i-3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{k-(i-1)} & -a_{k-(i-2)} & \cdots & -a_k & 0 & 0 & \cdots & 1 \\
\hline
1 & 0 & \cdots & 0 & -a_k & -a_{k-1} & \cdots & -a_{k-(i-1)} \\
-a_1 & 1 & \cdots & 0 & 0 & -a_k & \cdots & -a_{k-(i-2)} \\
-a_2 & -a_1 & \cdots & 0 & 0 & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{i-1} & -a_{i-2} & \cdots & 1 & 0 & 0 & \cdots & -a_k
\end{array}\right| \tag{17}$$

in which determinant order $i = 1, 2, 3, \ldots, k$.

The system is stable if and only if $\Delta_i \leq 0$ for even values of $i$, and $\Delta_i \geq 0$ for odd values of $i$.

In the case of the third-order predictor ($k = 3$), the stability region $S_3$ is described by the following conditions:

$$\Delta_1 = \begin{vmatrix} -a_3 & 1 \\ 1 & -a_3 \end{vmatrix} \le 0,$$

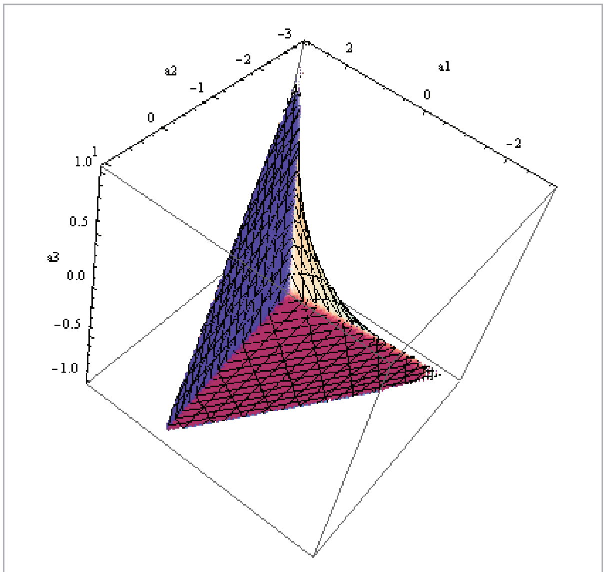
$$\Delta_2 = \begin{vmatrix} -a_3 & 0 & 1 & -a_1 \\ -a_2 & -a_3 & 0 & 1 \\ 1 & 0 & -a_3 & -a_2 \\ -a_1 & 1 & 0 & -a_3 \end{vmatrix} \ge 0, \tag{18}$$

$$\Delta_3 = \begin{vmatrix} -a_3 & 0 & 0 & 1 & -a_1 & -a_2 \\ -a_2 & -a_3 & 0 & 0 & 1 & -a_1 \\ -a_1 & -a_2 & -a_3 & 0 & 0 & 1 \\ 1 & 0 & 0 & -a_3 & -a_2 & -a_1 \\ -a_1 & 1 & 0 & 0 & -a_3 & -a_2 \\ -a_2 & -a_1 & 1 & 0 & 0 & -a_3 \end{vmatrix} \le 0.$$

The stability region $S_3$ is shown in Fig. 2.

**Figure 2**

The stability region $S_3$ of the third-order prediction filter




The probability density function is:

$$f_3(a_1, a_2, a_3) = \prod_{i=1}^{3} \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left[ -\frac{1}{2} \sum_{i=1}^{3} \left( \frac{a_i - \bar{a}_i}{\sigma_i} \right)^2 \right] \tag{19}$$

Theoretical value for the probability of stability is given by:

$$P_{S_3} = \frac{\iiint\limits_{S_3} f_3(a_1, a_2, a_3)\, da_1 da_2 da_3}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_3(a_1, a_2, a_3) da_1 da_2 da_3}. \tag{20}$$

Hence, we can see that already for the third order, calculation of the probability of stability becomes very complex. This is the reason why we use the Monte Carlo method for this purpose.

An experiment for obtaining predictor coefficients values was performed for recorded speech signal of 10200 samples with sampling frequency of 8KHz and resolution 16 bit/sample. The available signal was divided into frames of length $M$ , and for each frame, optimal values of predictor coefficients were calculated using the adaptive differential pulse code modulation (ADPCM) method [16]. The experiment was repeated for each frame length and the PDF of predictor coefficients $a_i$ , mean values $\bar{a}_i$ , and standard deviations $\sigma_i$ ( $i = 1, 2, 3$ ), were calculated.

Now, we can perform stability estimation of the third-order prediction filter. According to the results in the previous section (related to the accuracy of the Monte Carlo method), we perform all Monte Carlo simulation experiments with $N_3 = 10^7$ trials. We generated a simple code in the *Matlab* software package for Monte Carlo 3D numerical integration. We calculate the ratio between favourable cases (the predictor coefficients values which satisfy (18)) and the total number of trials, in order to obtain the desired values. These values of the probability of stability for five different frame lengths are given in Table 2.

**Table 2**

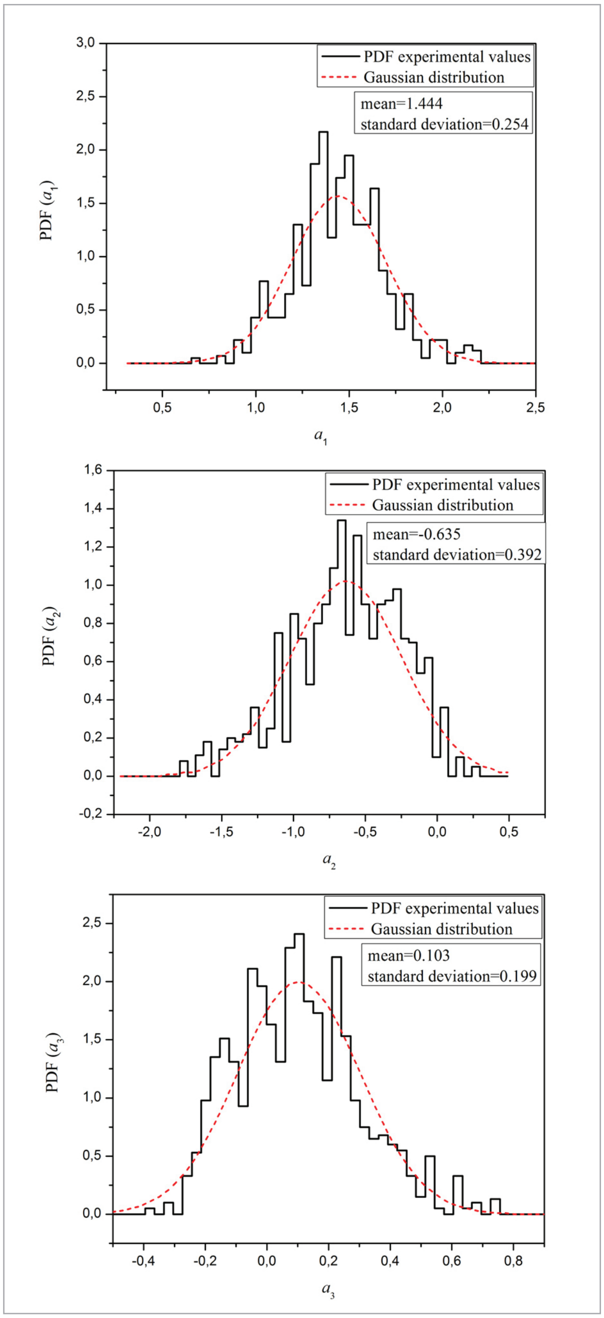
Probabilities of stability of the third-order prediction filter for different values of frame length $M$

| $M$ [samples] | 10 | 20 | 50 | 100 | 150 |
|---|---|---|---|---|---|
| $\bar{a}_1$ | 0.983 | 1.138 | 1.316 | 1.444 | 1.510 |
| $\sigma_1$ | 0.237 | 0.246 | 0.258 | 0.254 | 0.260 |
| $\bar{a}_2$ | -0.191 | -0.289 | -0.462 | -0.635 | -0.718 |
| $\sigma_2$ | 0.276 | 0.338 | 0.385 | 0.392 | 0.410 |
| $\bar{a}_3$ | -0.037 | -0.003 | 0.042 | 0.103 | 0.126 |
| $\sigma_3$ | 0.156 | 0.179 | 0.192 | 0.199 | 0.205 |
| $P$ ($N_3$=$10^7$) | **0.718** | **0.584** | **0.477** | **0.411** | **0.391** |

Distributions of coefficients $a_1$, $a_2$, and $a_3$ for $M = 20$, are shown in Fig. 3 for the illustrative purpose. Normal (Gaussian) distribution with the same mean and standard deviations is also shown for comparison.

**Figure 3**

The probability density function of predictor coefficients $a_1$, $a_2$, and $a_3$, respectively (for $M = 20$)




Now, in order to demonstrate the effectiveness of the proposed method, we perform some experiments with approximate method for probability of stability estimation given in [6, 7, 27]. By using two theorems given and proven in [7, 27], two stability regions, $\overline{S_k}$ and $\underline{S_k}$, described with appropriate relations are given. The stability region $S_k$ is limited above and below with these two regions, i.e. $\underline{S_k} \in S_k \in \overline{S_k}$. It means that we can calculate upper and lower limit values of probability of stability. In this paper, we calculate stochastic stability, so application of this approximate method can also become complex although the regions which bound $S_k$ are much easier for integration. Because of that, the following expressions are given in [27]:

$$P_{\overline{S_k}} = \left(-\frac{1}{2}\right)^k \prod_{i=1}^{k} \left( \Phi\left( \frac{-\binom{k}{i} - \overline{a_i}}{\sqrt{2\sigma_i}} \right) - \Phi\left( \frac{\binom{k}{i} - \overline{a_i}}{\sqrt{2\sigma_i}} \right) \right) \quad (21)$$

$$P_{\underline{S_k}} = \left(-\frac{1}{2}\right)^k \prod_{i=1}^{k} \left( \Phi\left( \frac{-\frac{1}{n} - \overline{a_i}}{\sqrt{2\sigma_i}} \right) - \Phi\left( \frac{\frac{1}{n} - \overline{a_i}}{\sqrt{2\sigma_i}} \right) \right) \quad (22)$$

where $\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-z^2)\, dz$ is the Laplace function, and $P_{\overline{S_k}}$ and $P_{\underline{S_k}}$ are the probabilities of the upper and lower bounds of the stability region $S_k$, respectively, i.e.: $P_{\underline{S_k}} \le P_{S_k} \le P_{\overline{S_k}}$.

In [6, 7], the value for probability of stability is approximated with the upper bound, i.e.: $P_{S_k} \approx P_{\overline{S_k}}$. Now, we can calculate the approximate values for probabilities using these relations. For $M = 20$, e.g., and given parameters (see Table 2), we obtain the following probabilities: $P_{\overline{S_3}} = 0.785$ and $P_{\underline{S_3}} = 2 \cdot 10^{-4}$. Hence, using the proposed method [27] we do not have enough information about the exact value of probability, but we only know that it is between 0.0002 and 0.785. The range is very wide, unfortunately. For other frame lengths, the proposed approximation is also very rough and conclusion is the same.

**Remark 2.** Despite the fact that there is no need for alternative methods in the case of the lower-order prediction filters, a comparison can be made, too. In the case of the second-order predictor and previously given values, we obtain $P_{\overline{S_2}} = 0.8842$ and $P_{\underline{S_2}} < 0.0001$

by using approximations (21) and (22). The range is too wide again. For the second-order prediction filter we have exact value for probability of stability 0.6556. If we approximate probability with upper value 0.8842 as it is proposed in [6, 7], the error is 0.2286, i.e. much larger than using the Monte Carlo method (Table 1).

In the case of the prediction filter with the fourth-order predictor, the probability of stability estimation is performed in the similar way. The experiments were performed for the same signal sample and the same frame lengths ( $M = 10, 20, 50, 100, 150$ ).

We also use the Schur–Cohn stability criterion to obtain the stability region for the fourth-order predictor, $S_4$ (17).

The probability density function for the fourth-order system is:

$$f_4(a_1, a_2, a_3, a_4) = \prod_{i=1}^{4} \frac{1}{\sigma_i \sqrt{2\pi}} \times \\ \times \exp\left[ -\frac{1}{2} \sum_{i=1}^{4} \left( \frac{a_i - \bar{a}_i}{\sigma_i} \right)^2 \right]. \tag{23}$$

Theoretical value for probability of stability is given by:

$$P_{S_4} = \frac{\iiiint\limits_{S_4} f_4(a_1, a_2, a_3, a_4) da_1 da_2 da_3 da_4}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_4(a_1, a_2, a_3, a_4) da_1 da_2 da_3 da_4} \tag{24}$$

By using the Monte Carlo simulation experiment with the same number of trials ( $N_4 = 10^7$ ), we obtain values for the probability of stability given in Table 3.

**Table 3**

Probabilities of stability of the fourth-order prediction filter for different values of frame length $M$

| $M$ [sample] | 10 | 20 | 50 | 100 | 150 |
|---|---|---|---|---|---|
| $\bar{a}_1$ | 0.980 | 1.140 | 1.340 | 1.474 | 1.540 |
| $\sigma_1$ | 0.246 | 0.257 | 0.283 | 0.269 | 0.275 |
| $\bar{a}_2$ | -0.203 | -0.318 | -0.543 | -0.763 | -0.860 |
| $\sigma_2$ | 0.312 | 0.396 | 0.506 | 0.501 | 0.507 |
| $\bar{a}_3$ | 0.035 | 0.105 | 0.233 | 0.377 | 0.430 |
| $\sigma_3$ | 0.213 | 0.272 | 0.374 | 0.363 | 0.388 |
| $\bar{a}_4$ | -0.074 | -0.094 | -0.142 | -0.194 | -0.211 |
| $\sigma_4$ | 0.133 | 0.141 | 0.172 | 0.170 | 0.193 |
| P ($N_4$=$10^7$) | **0.648** | **0.480** | **0.301** | **0.263** | **0.231** |

**Remark 3.** Using approximations (21) and (22), we obtain upper and lower bounds for probability (e.g. for $M = 20$ ) 0.69604 and 0.00003, respectively.

The proposed method can be easily applicable for any higher-order predictor, where classical integration cannot be applied and other methods give much bigger errors than the Monte Carlo method.

## Conclusion

In this paper, we performed the stability analysis of DPCM prediction filters with the higher-order predictors. We calculated the probability of stability values. Because of very complex classical numerical integration, we proposed the Monte Carlo integration.

We used the method, which was previously verified (for the first and second order), for the probability of stability estimation for the third- and fourth-order prediction filters.

The proposed method can be applied in the same way to the higher-order predictors, where classical methods become more and more complex. Experiments were performed for different number of trials for better accuracy.

Hence, the Monte Carlo method allows us to easily determine the probability of stability of the prediction filter with arbitrary order predictor, even when the classical integration method is not applicable. A problem with the error which occurs during the Monte Carlo integration is solved by increasing the number of trials in experiments, depending on the required accuracy set in advance. This is not possible for already developed methods for higher-order systems.

## Acknowledgment

This work was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia for the period 2011-2017 (projects: III 43007, III 44006, and TR 35005).

## References

1. Antić, D., Jovanović, Z., Danković, N., Spasić, M., Stankov, S. Probability estimation of certain properties of the imperfect systems. Proceedings of IEEE 7th International Symposium on Applied Computational Intelligence and Informatics, (SACI 2012), Timisoara, Romania, May 24-26, 2012, 213-216. http://dx.doi.org/10.1109/SACI.2012.6250004
2. Antić, D., Jovanović, Z., Danković, N., Spasić, M., Stankov, S. Probability estimation of defined properties of the real technical systems with stochastic parameters. Scientific Bulletin of the "Politehnica" University of Timişoara, Romania, Transactions on Automatic Control and Computer Science, 2012, 57(71-2), 67-74.
3. Antić, D., Nikolić, S., Milojković, M., Danković, N., Jovanović, Z., Perić S. Sensitivity analysis of imperfect systems using almost orthogonal filters. Acta Polytechnica Hungarica, 2011, 8(6), 79-94.
4. Benesty, J., Sondhy, M., Huang, Y. Introduction to speech processing. In: Springer Handbook of Speech Processing, Eds. Benesty, J., Mohan Sondhi, M., Huan, Y. A., Springer, Chapter 7, 2008. http://dx.doi.org/10.1007/978-3-540-49127-9_7
5. Combettes, P. L., Trussell, H. J. Stability of the linear prediction filters: A set theoretic approach. IEEE International Conference on Acoustics, Speech, and Signal Processing, New York, USA, April 11-14, 1988, 4, 2288-2291. http://dx.doi.org/10.1109/ICASSP.1988.197094
6. Danković, B., Jovanović, Z. On the reliability of discrete-time control systems with random parameters. Quality technology and Quantitative Management, 2005, 2(1), 53-63. http://dx.doi.org/10.1080/16843703.2005.11673080
7. Danković, B., Vidojković, B. M., Vidojković, B. The probability stability estimation of discrete-time systems with random parameters. Control and Intelligent Systems, 2007, 35(2), 134-139. http://dx.doi.org/10.2316/Journal.201.2007.2.201-1618
8. Danković, N. B., Perić, Z. H. A probability of stability estimation of DPCM system with the first order predictor. Facta Universitatis, Series: Automatic Control and Robotics, 2013, 12(2), 131-138.
9. Danković, N., Perić, Z., Antić, D., Mitić, D., Spasić, M. On the sensitivity of the recursive filter with arbitrary order predictor in DPCM system, Serbian Journal of Electrical Engineering, 2014, 11(4), 609-616. http://dx-.doi.org/10.2298/SJEE1404609D
10. Fira, C., Goras, L. An ECG signals compression method and its validation using NNs. IEEE Transactions on Biomedical Engineering, 2008, 55(4), 1319-1326. [Online]. Available: http://dx.doi.org/10.1109/TBME.2008.918465
11. Fishman, G. S. Monte Carlo: Concepts, algorithms, and applications. Springer, New York, 1996. http://dx.doi.org/10.1007/978-1-4757-2553-7
12. Gentle, J. E. Random Number Generation and Monte Carlo Methods. Statistics and Computing, 2nd edition, Springer, New York, 1998. http://dx.doi.org/10.1007/b97336
13. Goldstein, L. H., Liu, B. Deterministic and stochastic stability of adaptive differential pulse code modulation. IEEE Transactions on Information Theory, 1977, 23(4), 445-453. http://dx.doi.org/10.1109/TIT.1977.1055754
14. Gould, H., Tobochnik, J., Christian, W. An Introduction to Computer Simulation Methods, 3rd edition. Chapter 12, 2011.
15. Jalaleddine, S. M. S., Hutchens, C. G., Strattan, R. D., Coberly, W. ECG data compression techniques-a unified approach. IEEE Transactions on Biomedical Engineering, 1990, 37(4), 329-343. http://dx.doi.org/10.1109/10.52340
16. Jayant, N. S., Noll, P. Digital Coding of Waveforms, Principles and Applications to Speech and Video. Prentice Hall, Chapter 6, 1984.
17. Jovanović, Z., Danković, N., Spasić, M., Stankov, S., Icić, Z. Practical testing of the probability of stability of the discrete system with random parameters. In: Proceedings of 10th International Conference on Applied Electromagnetics, (ΠΕС 2011), Niš, Serbia, Sept. 25-29, 2011.
18. Kailath, T. A Theorem of I. Schur and its impact on modern signal processing, operator theory: Advances and applications. In: I. Schur Methods in Operator Theory and Signal Processing, Ed. I. Gohberg, 1986, Birkhäuser Basel, 18, 9-30. http://dx.doi.org/ 10.1007/978-3-0348-5483-2_2
19. Macchi, O., Uhl, C. The stability of DPCM transmission system. IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing, 1992, 39(10), 705-722. http://dx.doi.org/10.1109/82.199897
20. Mitchell, H. B., Estrakh, D. D. A modified OWA operator and its use in Lossless DPCM image compression. International Journal of Uncertainty, Fuzziness and

Knowledge-Based Systems, 1997, 5(4), 429-436. http://dx.doi.org/10.1142/S0218488597000324

21. Perić, Z., Denić, D., Nikolić, J., Jocić, A., Jovanović, A. DPCM quantizer adaptation method for efficient ECG signal compression. Journal of Communications Technology and Electronics, 2013, 58(12), 1241-1250. http://dx.doi.org/10.1134/S1064226913130068
22. Perić, Z., Jocić, A., Nikolić, J., Velimirović, L., Denić D. Analysis of differential pulse code modulation with forward adaptive Lloyd-Max's quantizer for low bit-rate speech coding. Revue Roumanie des Sciences Techniques, 2013, 58(4), 424-434. http://www.revue.elth.pub.ro/index.php?action=main&year=2013&issue=4. Accessed on December 20, 2015.
23. Shlien, S. A geometric description of stable linear predictive coding digital filters. IEEE Transactions on Information Theory, 1985, 31(4), 545-548. http://dx.doi.org/10.1109/TIT.1985.1057070
24. Taralova, I., Fournier-Prunaret, D. Dynamical study of a second order DPCM transmission system modeled by a piece-wise linear function. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 2002, 49(11), 1592-1609. http://dx.doi.org/10.1109/TCSI.2002.804592
25. Uhl, C., Macchi, O. The stability of a DPCM transmission system with an order t predictor. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 1993, 40(1), 50-55. http://dx.doi.org/10.1109/81.215342
26. Vidojković, B., Jovanović, Z., Milojković, M. The probability stability estimation of the system based on the quality of the components. Facta Universitatis, Series: Electronics and Energetics, 2006, 19(3), 385-391. http://dx.doi.org/10.2298/FUEE0603385V
27. Zlatković, B. M., Samardžić, B. One way for the probability of stability estimation of discrete systems with randomly chosen parameters. IMA Journal of Mathematical Control and Information, 2012, 29(3), 329-341. http://dx.doi.org/10.1093/imamci/dnr041
28. Zschunke, W. DPCM picture coding with adaptive prediction. IEEE Transactions on Communications, 1977, 25(11), 1295-1302. http://dx.doi.org/10.1109/TCOM.1977.1093771

## Summary / Santrauka

This paper presents the stability analysis of the linear recursive (prediction) filters with higher-order predictors in a DPCM (differential pulse-code modulation) system, where traditional methods become too difficult and complex. Stability conditions for the third- and fourth-order predictor are given by using the Schur-Cohn stability criterion. The probability of stability estimation is performed by using the Monte Carlo method. Verification of the proposed method is performed for lower-order predictors (the first- and second-order). We calculated numerical values of the probability of stability for higher-order predictors and previously experimentally obtained parameters. With large enough number of trials (samples) in Monte Carlo simulation, we reach the desired accuracy.

Straipsnyje pateikta tiesinių rekursinių (numatymų) filtrų su aukštesnės eilės prediktoriais stabilumo analizė diferencinio pulso-kodo moduliacijos (angl. *differential pulse-code modulation (DPCM)*) sistemoje, kurioje tradiciniai metodai yra per daug sudėtingi. Stabilumo sąlygos trečiosios ir ketvirtosios eilės prediktoriui užtikrinamos taikant Schuro ir Cohno stabilumo kriterijų. Stabilumo įvertinimo tikimybė apskaičiuota taikant Monte Karlo metodą. Siūlomo metodo verifikacija atlikta žemesnės (pirmosios ir antrosios) eilės prediktoriams. Apskaičiuotos skaitinės stabilumo tikimybės reikšmės aukštesnės eilės prediktoriams ir anksčiau eksperimentiškai gautų parametrų įverčiai. Atlikus gana daug bandymų (imčių) Monte Karlo simuliacijoje pavyko pasiekti norimą tikslumą.