

## NONLINEAR PHENOMENA IN DIGITAL NETWORKS

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**Abstract.** The paper is devoted to the analysis of queueing systems in the context of the network and communications theory. We investigate a theorem on the law of the iterated logarithm for a queue of customers in an open queueing network and its applications to the mathematical models of the Negative ACKnowledgement and Internet systems.

**Key words:** Models of information systems, open queueing network, queue of customers, law of the iterated logarithm.

### 1. STATEMENT OF THE PROBLEM

The paper is devoted to the analysis of queueing systems in the context of the network and communications theory. We investigate a theorem on the law of the iterated logarithm (LIL) for the queue of customers in an open queueing network and its applications to the mathematical models of the Negative ACKnowledgement (NACK) and Internet systems.

Now we try to outline the research of the LIL in queueing systems. In [2], Bingham familiarizes with the general theory on the LIL and its numerous applications in various fields of probability theory. The main part in the development of the theory on the LIL was played by Strassen [24], where the functional variant of the LIL for a Wiener process was proved. The paper of Iglehart [9] can be considered as the first work on the LIL in the queueing theory. Applying the approach of Strassen [24], Iglehart proved the LIL in [9] under the conditions of heavy traffic for the queue of customers, waiting time of a customer, a virtual waiting time of a customer, and other important probability characteristics of the classical queueing system  $G1/G/1$  and more general systems (e.g., a multiple queueing system). Also, a functional variant of the LIL for a renewal process was proved in his work [9]. Using the results of Iglehart [10] and [11], the survey of Whitt [25] presents the proof of theorems on the LIL for the waiting time of a customer, the occupation time process, and the extreme value of the waiting time of a customer in the queueing system  $G1/G/1$ . The works of Glynn and Whitt [7, 8] pre-

sent the proof of theorems on the LIL for a cumulative process associated with the queue of customers and waiting time of a customer in an ordinary queueing system  $G1/G/1$ .

We note that the research of the LIL in more general systems than the queueing system  $G1/G/1$  or multiphase queueing systems has just started (see the article of Asmussen [1]). In the papers of Minkevičius [15] and [16] the LIL is proved for the queue of customers, the waiting time of a customer, a virtual waiting time of a customer in heavy traffic in a multiphase queueing system. The work of Sakalauskas and Minkevičius [23] also gives the proof of the theorem on the LIL under the conditions of heavy traffic for a virtual waiting time of a customer in the open Jackson network.

Now we try to consider the papers on a queue in heavy traffic conditions. In the paper of Chen, Xinyang and Yao [5] a semi-martingale reflected Brownian motion approximation is developed for the performance processes such as workload, queue, and sojourn time. In the paper of Massey and Srinivasan [14], the steady-state distribution of the queue process, using tensor and Kronecker products, shows that it is of the matrix-geometric structure. Dai and Dai in [6] proved that an appropriately normalized queue process converges in distribution to a  $d$ -dimensional reflecting Brownian motion under the heavy traffic condition. Puhalskii in [20] established moderate-deviation principles for the queue, virtual waiting time and sojourn processes. In [12], Yamada has showed that the normalized queue processes at the nodes converge in distribution to a reflected, multi-

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variate diffusion process whose drift and diffusion coefficients are state dependent and nonsingular. In the article of Kushner and Martins [13], the authors study the pathwise average cost per unit time problems for controlled and uncontrolled open queueing networks in heavy traffic. In the paper of Zhang Hanqin and Xu Guang-hui [26] strong approximations for an open queueing network in heavy traffic are proved. Peterson [19] has proved that, under heavy traffic conditions, the vector processes of total unfinished workloads converge to a multidimensional regulated Brownian motion. In the article of Reiman and Simon [22], the authors consider an open queueing network with multiple classes, priorities, "arbitrary" routing, and general service time distribution. Using a heavy traffic limit theorem for open queueing networks, Reiman in [21] found the correct diffusion approximation for sojourn times in Jackson networks with a single-server station. As one can see, there are only several works designed to explore a queue in a more complicated than the classical single-server queue: tandem, multiphase queue, open queueing network (see the articles of Boxma [3, 4], Zhang Hanqin and Xu Guang-hui [26], Massey and Srinivasan [14], and Sakalauskas and Minkevičius [23]).

In this paper, we investigate an open queueing network model in heavy traffic. We present the LIL for the queue of customers in an open queueing network. The main tool for the analysis of these queueing systems in heavy traffic is a functional LIL for the renewal process (the proof can be found in [24] and [9]).

The service discipline is "first come, first served" (FCFS). We consider open queueing networks with the FCFS service discipline at each station and general distributions of interarrival and service times. We study the queueing network with  $k$  single server stations, each of which has an associated infinite capacity waiting room. Every station has an arrival stream from outside the network, and the arrival streams are assumed to be mutually independent renewal processes. Customers are served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. Service times and routing decisions form mutually independent sequences of independent identically distributed random variables.

The basic components of the queueing network are arrival processes, service processes, and routing processes. In particular, there are mutually independent sequences of independent identically distributed random variables  $\{z_n^{(j)}, n \geq 1\}$ ,  $\{S_n^{(j)}, n \geq 1\}$  and  $\{\Phi_n^{(j)}, n \geq 1\}$  for  $j = 1, 2, \dots, k$ . The random vari-

ables  $z_n^{(j)}$  and  $S_n^{(j)}$  are strictly positive, and  $\Phi_n^{(j)}$  has support in  $\{0, 1, 2, \dots, k\}$ . We define  $\mu_j = \left(M \left[S_n^{(j)}\right]\right)^{-1} > 0$ ,  $\sigma_j = D \left(S_n^{(j)}\right) > 0$ ,  $\lambda_j = \left(M \left[z_n^{(j)}\right]\right)^{-1} > 0$ , and  $a_j = D \left(z_n^{(j)}\right) > 0$ ,  $j = 1, 2, \dots, k$ ; all of these sequences are assumed to be finite. We denote  $p_{ij} = P \left(\Phi_n^{(i)} = j\right) > 0$ ,  $i, j = 1, 2, \dots, k$ . In the context of the queueing network, the random variables  $z_n^{(j)}$  function as interarrival times (from outside the network) at the station  $j$ , while  $S_n^{(j)}$  is the  $n$ th service time at station  $j$ , and  $\Phi_n^{(j)}$  is a routing indicator for the  $n$ th customer served at the station  $j$ . If  $\Phi_n^{(i)} = j$  (which occurs with probability  $p_{ij}$ ), then the  $n$ th customer served at the station  $i$  is routed to the station  $j$ . When  $\Phi_n^{(i)} = 0$ , the associated customer leaves the network. To construct renewal processes generated by the interarrival and service times, we assume  $z_j(0) = 0, z_j(l) = \sum_{m=1}^l z_m^{(j)}, S_j(0) = 0, S_j(l) = \sum_{m=1}^l S_m^{(j)}, l \geq 1, j = 1, 2, \dots, k$ . We now define  $a_j(t) = \max(l \geq 0 : z_j(l) \leq t)$ ,  $x_j(t) = \max(l \geq 0 : S_j(l) \leq t)$ ,  $\tilde{\tau}_j(t)$  as the total number of customers routed to the  $j$ th station of the network until time  $t$ ,  $\tau_j(t)$  as the total number of customers after service departure from the  $j$ th station of the network until time  $t$ ,  $\tau_{ij}(t)$  as the total number of customers after service departure from the  $i$ th station of the network and routed to the  $j$ th station of network until time  $t$ ,  $p_{ij}^t = \frac{\tau_{ij}(t)}{\tau_i(t)}$  as part of the total number of customers which, after service at the  $i$ th station of the network, are routed to the  $j$ th station of the network,  $i, j = 1, 2, \dots, k$  and  $t > 0$ . Note that this system is quite general, encompassing the tandem system, acyclic networks of  $GI/G/1$  queues, networks of  $GI/G/1$  queues with feedback, and an open queueing network.

First, let us denote by  $Q_j(t)$  the queue of customers at the  $j$ th station of the queueing network at time  $t$ ;  $\beta_j = \lambda_j + \sum_{i=1}^k \mu_i \cdot p_{ij} - \mu_j > 0$ ,  $\hat{\sigma}_j^2 = (\lambda_j)^3 \cdot a_j + \sum_{i=1}^k (\mu_i)^3 \cdot \sigma_i \cdot (p_{ij})^2 + (\mu_j)^3 \cdot \sigma_j > 0$ ,  $j = 1, 2, \dots, k$ .

Suppose that the queue of customers in each station of the open queueing network is unlimited. All random variables are defined on one common probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ .

We assume that the following conditions are ful-

filled:

$$\lambda_j + \sum_{i=1}^k \mu_i \cdot p_{ij} > \mu_j, \quad j = 1, 2, \dots, k. \quad (1)$$

Note that these conditions guarantee that there exists a queue of customers and it is constantly growing.

## 2. The Main Result

One of the results of the paper is the following theorem on the LIL for the queue of customers in an open queueing network.

**Theorem 1.** *If conditions (1) are fulfilled, then*

$$P \left( \overline{\lim}_{t \rightarrow \infty} \frac{Q_j(t) - \beta_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = 1 \right) = 1, \quad j = 1, 2, \dots, k$$

$$\text{and } a(t) = \sqrt{2t \ln \ln t}.$$

*Proof.* First define  $\hat{x}_j(t) = \sum_{i=1}^k x_i(t) \cdot p_{ij} + a_j(t) - x_j(t)$ ,  $w_j(t) = x_j(t) \cdot \left\{ \sum_{i=1}^k |p_{ij}^t - p_{ij}| \right\}$ ,  $\gamma_j(t) = \sup_{0 \leq s \leq t} (x_j(s) - \tau_j(s))$ ,  $j = 1, 2, \dots, k$  and  $t > 0$ .

By definition of the queue of customers at the stations of the network, we get that

$$\begin{aligned} Q_j(t) &= \tilde{\tau}_j(t) - \tau_j(t) = \tilde{\tau}_j(t) - x_j(t) + x_j(t) \\ &- \tau_j(t) \leq \tilde{\tau}_j(t) - x_j(t) + \sup_{0 \leq s \leq t} (x_j(s) - \tau_j(s)) \\ &= \sum_{i=1}^k \tau_i(t) \cdot p_{ij}^t + a_j(t) - x_j(t) + \sup_{0 \leq s \leq t} (x_j(s) - \tau_j(s)) \\ &\leq \sum_{i=1}^k x_i(t) \cdot |p_{ij}^t - p_{ij}| + \sup_{0 \leq s \leq t} (x_j(s) - \tau_j(s)) \\ &\leq \hat{x}_j(t) + \sum_{i=1}^k w_i(t) + \sup_{0 \leq s \leq t} (x_j(s) - \tau_j(s)), \end{aligned} \quad (2)$$

$j = 1, 2, \dots, k$  and  $t > 0$ .

This implies that

$$Q_j(t) \leq \hat{x}_j(t) + \gamma_j(t) + \sum_{i=1}^k w_i(t), \quad (3)$$

$$j = 1, 2, \dots, k \text{ and } t > 0.$$

Also, note that

$$\begin{aligned} Q_j(t) &\geq \tilde{\tau}_j(t) - x_j(t) = \sum_{i=1}^k \tau_i(t) \cdot p_{ij}^t \\ &+ a_j(t) - x_j(t) = \sum_{i=1}^k x_i(t) \cdot p_{ij}^t + a_j(t) \\ &- x_j(t) + \sum_{i=1}^k (\tau_i(t) - x_i(t)) \cdot p_{ij}^t \\ &= \sum_{i=1}^k x_i(t) \cdot p_{ij} + a_j(t) - x_j(t) \\ &+ \sum_{i=1}^k x_i(t) \cdot (p_{ij}^t - p_{ij}) \\ &= \hat{x}_j(t) + \sum_{i=1}^k x_i(t) \cdot (p_{ij}^t - p_{ij}) \\ &+ \sum_{i=1}^k (\tau_i(t) - x_i(t)) \cdot p_{ij}^t \\ &\geq \hat{x}_j(t) - \sum_{i=1}^k x_i(t) \cdot |p_{ij}^t - p_{ij}| \\ &- \sum_{i=1}^k (x_i(t) - \tau_i(t)) \cdot p_{ij}^t \geq \hat{x}_j(t) \\ &- \sum_{i=1}^k w_i(t) - \sum_{i=1}^k (x_i(t) - \tau_i(t)) \\ &\geq \hat{x}_j(t) - \sum_{i=1}^k w_i(t) \\ &- \sup_{0 \leq s \leq t} \sum_{i=1}^k (x_i(t) - \tau_i(t)) \\ &\geq \hat{x}_j(t) - \sum_{i=1}^k w_i(t) - \sum_{i=1}^k \gamma_i(t), \end{aligned} \quad (4)$$

$$j = 1, 2, \dots, k \text{ and } t > 0.$$

Hence it follows that

$$Q_j(t) \geq \hat{x}_j(t) - \sum_{i=1}^k w_i(t) - \sum_{i=1}^k \gamma_i(t), \quad (5)$$

$j = 1, 2, \dots, k$  and  $t > 0$ . By combining (3) and (5), we can write

$$|Q_j(t) - \hat{x}_j(t)| \leq \sum_{i=1}^k w_i(t) + \sum_{i=1}^k \gamma_i(t), \quad (6)$$

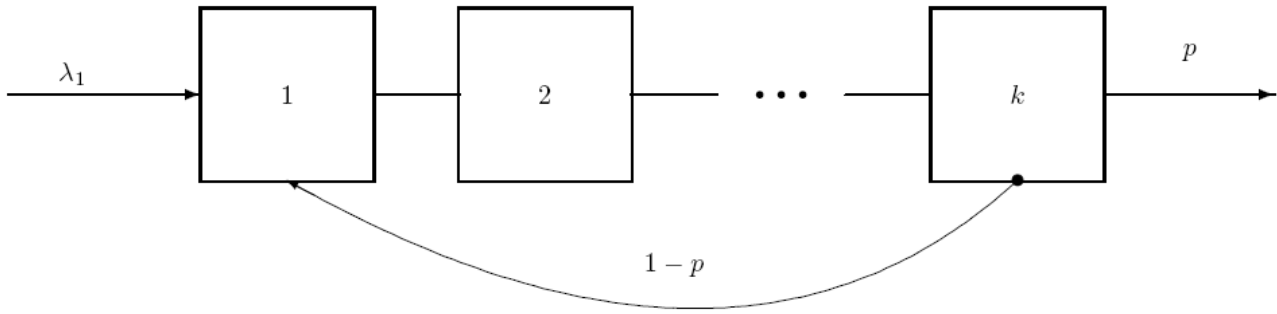


Figure 1. NACK-type switching system

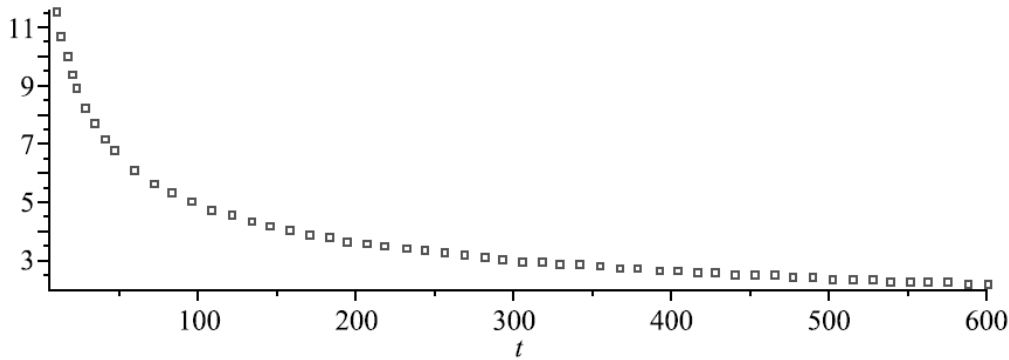


Figure 2. Values for  $\frac{M\tilde{Q}_1(t)}{t}$

$j = 1, 2, \dots, k$  and  $t > 0$ . The further proof is the same as in [18]. The proof of the theorem is complete.  $\square$

Note that inequality (6) is the key inequality to prove several laws (fluid approximations, functional limit theorems and LIL) for a queue of customers in open queueing networks in heavy traffic conditions.

### 3. On the model of the NACK-type switching facility

In this section, we consider a switching facility that transmits packages of data to a required destination (see Figure 1). A NACK is sent to the destination when a package has not been properly transmitted. In this case, the package in error is retransmitted as soon as the NACK has been received. We assume that the switching facility is composed of  $k$  nodes in series, each modelled as a  $G/GI/1$  queue with the common service rate  $\mu$ . In other words, we now have an open Jackson network with  $k$   $G/GI/1$  queues where  $\lambda_j = 0$  for  $j = 1, 2, 3, \dots, k$  (no external arrivals at nodes  $2, 3, \dots, k$ ),  $\mu_i = \mu$  for  $i = 2, 3, \dots, k$ ,

$p_{i,i+1} = 1$  for  $i = 1, 2, \dots, k - 1$ ,  $p_{k0} = p$  and  $p_{k1} = 1 - p$  (usually  $p = 0.9$ ).

Thus, we investigate a NACK-type switching system which consists of  $k$  service nodes and in which  $S_n^{(j)} = S_n$ ,  $j = 1, 2, \dots, k$  (the service process is identical in the phases of the system). We denote by  $\bar{Q}_j(t)$  the queue of packages in the  $j$ th phase of the NACK-type switching system at the time moment  $t$ ,  $j = 1, 2, \dots, k$  and  $t > 0$ .

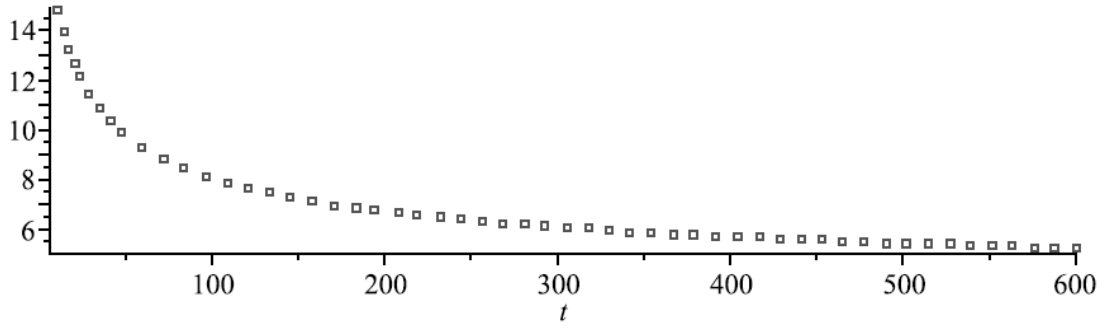
Let us define  $\bar{\beta}_1 = \lambda_1 - \mu \cdot (1 - p_{k1}) > 0$ ,  $\bar{\sigma}_1^2 = (\lambda_1)^3 \cdot a_1 + (\mu)^3 \cdot DS_n \cdot ((p_{k1})^2 + 1) > 0$ ,  $\bar{\beta}_j = \lambda_j - \mu \cdot (1 - p_{j-1j}) = 0$ ,  $\bar{\sigma}_j^2 = (\mu)^3 \cdot DS_n \cdot ((p_{j-1j})^2 + 1) = 2 \cdot (\mu)^3 \cdot DS_n > 0$ ,  $j = 2, 3, \dots, k$ .

Applying Theorem 2.1, we present a theorem and corollary about the queue of packages in the NACK-type switching system.

**Theorem 2.** *If conditions (1) are fulfilled, then*

$$P\left(\lim_{t \rightarrow \infty} \frac{\bar{Q}_j(t) - \bar{\beta}_j \cdot t}{\bar{\sigma}_j \cdot a(t)} = 1\right) = 1, \quad j = 1, 2, \dots, k.$$

**Corollary 1.** *If conditions (1) are fulfilled, then, for fixed  $\varepsilon > 0$ , there exists  $t(\varepsilon)$  such that for every  $t \geq$*



**Figure 3.** Values for  $\frac{M\bar{Q}_j(t)}{t}, j=2, 3, \dots, k$

$t(\varepsilon)$

$$(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t \leq \bar{Q}_j(t) \leq (1+\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t$$

$j = 1, 2, \dots, k.$

Evidently, Corollary 1 implies that, for fixed  $\varepsilon > 0$ , there exists  $t(\varepsilon)$  such that for every  $t \geq t(\varepsilon)$ ,

$$(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t \leq \bar{Q}_j(t) \leq (1+\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t,$$

where  $\varepsilon > 0, t > 0, j = 1, 2, \dots, k.$

Hence we can derive

$$(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t \leq M\bar{Q}_j(t) \leq (1+\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t, \quad |M(\bar{Q}_j(t) - \bar{\beta}_j \cdot t) - (1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t)| \leq 2 \cdot \varepsilon \cdot \bar{\sigma}_j \cdot a(t),$$

$$\left| M \left( \frac{\bar{Q}_j(t) - \bar{\beta}_j \cdot t}{\bar{\sigma}_j \cdot a(t)} \right) - (1-\varepsilon) \right| \leq 2 \cdot \varepsilon, \quad (7)$$

$j = 1, 2, \dots, k.$

Thus, it follows from (7) that

$$M\bar{Q}_j(t) \sim \bar{\beta}_j \cdot t + (1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t), \quad (8)$$

$j = 1, 2, \dots, k.$   $M\bar{Q}_j(t)$  is the average queue of packages in the NACK-type switching system at the time moment  $t, j = 1, 2, \dots, k$  and  $t > 0$ . We see from (8) that  $M\bar{Q}_j(t), j = 1, 2, \dots, k, t > 0$  consists of the linear function  $\bar{\beta}_j \cdot t$  and a nonlinear slowly increasing function  $(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t)$ .

Now we present an example from the network practice. Assume that packages of data (queries or messages) routed to the first device  $V_1$  at the rate  $\lambda_1$  of 21 per second during business hours. These packages are served at the rate  $\mu$  of 20 per second at the device  $V_1$ . After they have been served at the device  $V_1$ , the packages are routed to the second device  $V_2$ . Also, note that the packages are served at the rate  $\mu$  of 20 per second at the device  $V_2$ . So, the packages are

served in the devices  $V_1, V_2, \dots, V_k$ , and after they have been served at the device  $V_k$ , with the probability  $p = 0.9$  (probability that a package is received correctly) they leave the NACK-type system and are sent to the device  $V_1$  with probability  $1-p = 0.1$ .

Thus,  $DS_n = \frac{1}{\mu} = \frac{1}{20}, \lambda_1 = (M(z_n^{(1)}))^{-1} = 21, D(z_n^{(1)}) = \frac{1}{\lambda_1} = \frac{1}{21}, \bar{\beta}_1 = 3, \bar{\sigma}_1^2 = 845, \bar{\sigma}_1 = 29.0688, \bar{\beta}_j = 0, \bar{\sigma}_j^2 = 800, \bar{\sigma}_j = 28.2843, j = 2, 3, \dots, k, \varepsilon = 0.001, t \geq 10.$

Consequently,

$$M\bar{Q}_1(t) \sim \bar{\beta}_1 \cdot t + (1-\varepsilon) \cdot \bar{\sigma}_1 \cdot a(t) = (3) \cdot t + (29.0688) \cdot a(t). \quad (9)$$

From (9) we get (see Figure 2)

$$\frac{M\bar{Q}_1(t)}{t} = (3) + (29.0688) \cdot \sqrt{\frac{2 \ln \ln t}{t}}. \quad (10)$$

Similarly as in (10), we can obtain (see Figure 3)

$$\frac{M\bar{Q}_j(t)}{t} = (28.2843) \cdot \sqrt{\frac{2 \ln \ln t}{t}}, \quad (11)$$

$j = 2, 3, \dots, k.$

#### 4. On the model of the Internet linear network

In this section, we consider an Internet linear network with two resources and three routes (see Figure 4). We now assume that the linear network is composed of two nodes, each modelled as a  $G/GI/1$  queue. In other words, we have an open Jackson network with two  $G/GI/1$  queues where external arrivals at the first node are  $\lambda_{11}$  and  $\lambda_{12}$ . The packages are served in the first node with the rate  $\mu_1$ , afterwards Internet packages of data (queries or messages) with probability  $p_{12} = p$  (usually  $p = 0.1$ ) are sent to the

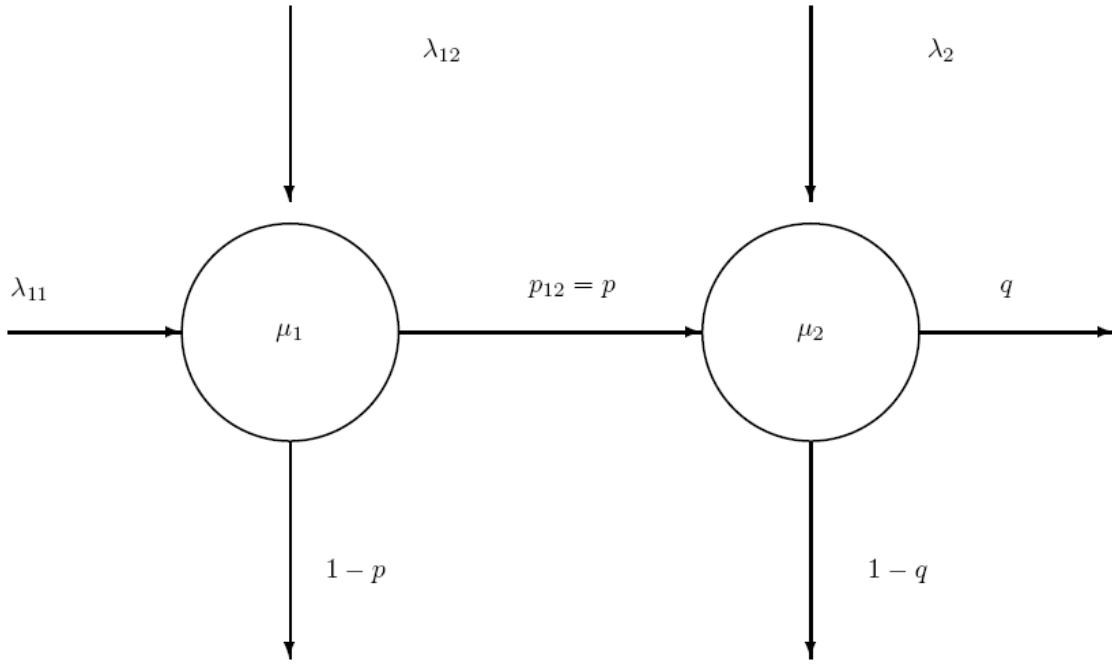


Figure 4. Model of the Internet linear network

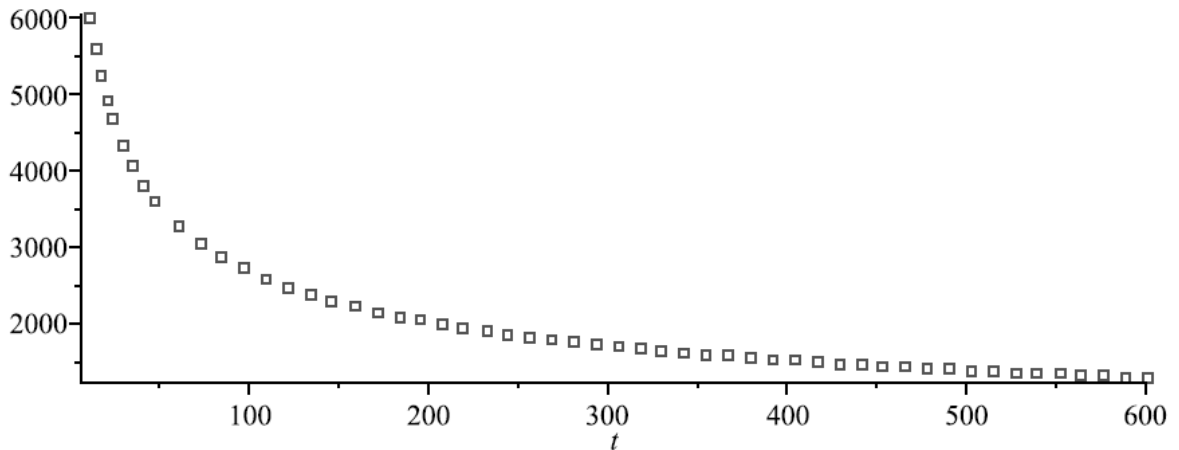


Figure 5. Values for  $\frac{M\tilde{Q}_1(t)}{t}$

second node and with probability  $p_{10} = 1 - p$  leave the system, an external route to the second node is  $\lambda_2$ . The packages are served in the second node with the rate  $\mu_2$ , then Internet packages, with probability  $p_{20} = q$  (usually  $q = 0.1$ ), leave the system in one direction, and with probability  $1 - q$  they leave the system in another direction. Next, denote by  $\tilde{Q}_j(t)$  the queue of Internet packages in the  $j$ th node of the Internet-type network at the time moment  $t$ ,  $j = 1, 2$  and  $t > 0$ . Define  $\tilde{\beta}_1 = \lambda_{11} + \lambda_{12} - \mu_1 > 0$ ,  $\tilde{\sigma}_1^2 = a_1 \cdot (\lambda_{11} + \lambda_{12})^3 + (\mu_1)^3 \cdot \sigma_1 > 0$ ,  $\tilde{\beta}_2 = \lambda_2 + \mu_1 \cdot p_{12} - \mu_2 > 0$ ,  $\tilde{\sigma}_2^2 = (\lambda_2)^3 \cdot a_2 + (\mu_1)^3 \cdot \sigma_1 \cdot$

$$(p_{12})^2 + (\mu_2)^3 \cdot \sigma_2 > 0.$$

Applying Theorem 2.1, we obtain the following theorem and corollary on the queue of packages in the Internet linear network system.

**Theorem 3.** *If conditions (1) are fulfilled, then*

$$P\left(\overline{\lim}_{t \rightarrow \infty} \frac{\tilde{Q}_j(t) - \tilde{\beta}_j \cdot t}{\tilde{\sigma}_j \cdot a(t)} = 1\right) = 1, \quad j = 1, 2.$$

**Corollary 1.** *If conditions (1) are fulfilled, then, for fixed  $\varepsilon > 0$ , there exists  $t(\varepsilon)$  such that for every  $t \geq$*

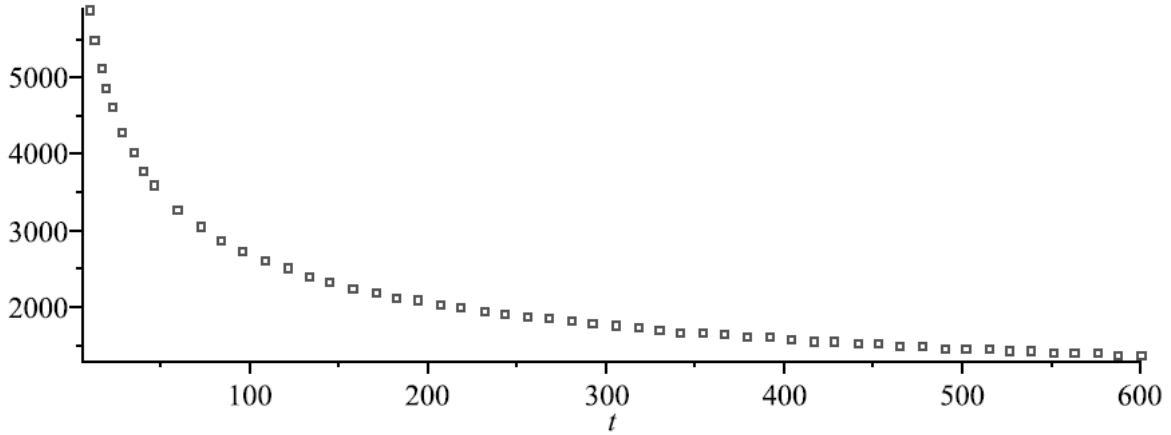


Figure 6. Values for  $\frac{M\tilde{Q}_2(t)}{t}$

$t(\varepsilon)$

$$\begin{aligned} (1 - \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t) + \tilde{\beta}_j \cdot t &\leq \tilde{Q}_j(t) \\ &\leq (1 + \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t) + \tilde{\beta}_j \cdot t, \quad j = 1, 2. \end{aligned}$$

Similarly as in (8) we can obtain

$$M\tilde{Q}_j(t) \sim \tilde{\beta}_j \cdot t + (1 - \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t), \quad j = 1, 2. \quad (12)$$

$M\tilde{Q}_j(t)$  is the average queue of packages in the Internet linear network at the time moment  $t$ ,  $j = 1, 2$  and  $t > 0$ . We see from (12) that  $M\tilde{Q}_j(t)$  consists of the linear function  $\tilde{\beta}_j \cdot t$  and a nonlinear slowly increasing function  $(1 - \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t)$ ,  $j = 1, 2$  and  $t > 0$ .

Now we give an example from the Internet network practice. Assume that packages are routed to the first node  $W_1$  at the rate  $\lambda_{11}$  of 5200 and  $\lambda_{12}$  of 5000 per second during business hours. These packages are served at the rate  $\mu_1$  of 10000 per second in the first node  $W_1$ . After service in the node  $W_1$  packages are routed to the second node  $W_2$  at the rates  $\lambda_{12} \cdot p$  of 1000 and  $\lambda_2$  of 9300 per second during business hours. Also note that the packages are served at the rate  $\mu_2$  of 10000 per second in the node  $W_2$ . After the packages have been served in the node  $W_2$ , they leave the system.

So,  $\tilde{\beta}_1 = 200$ ,  $\tilde{\sigma}_1^2 = 204040$ ,  $\tilde{\sigma}_1 = 14284.2$ ,  $\tilde{\beta}_2 = 300$ ,  $\tilde{\sigma}_2^2 = 187430$ ,  $\tilde{\sigma}_2 = 13690.5$ . Thus,

$$\begin{aligned} M\tilde{Q}_1(t) &\sim \tilde{\beta}_1 \cdot t + (1 - \varepsilon) \cdot \tilde{\sigma}_1 \cdot a(t) \\ &= (200) \cdot t + (14284.2) \cdot a(t). \end{aligned} \quad (13)$$

From (13) we get (see Figure 5)

$$\frac{M\tilde{Q}_1(t)}{t} = (200) + (14284.2) \cdot \sqrt{\frac{2 \ln \ln t}{t}}. \quad (14)$$

Similarly as in (14) we can obtain (see Figure 6)

$$\frac{M\tilde{Q}_2(t)}{t} = (300) + (13690.5) \cdot \sqrt{\frac{2 \ln \ln t}{t}}. \quad (15)$$

*Remark 1.* When modelling an Internet network system, we apply an heuristic argument, - in real conditions, an average Internet network system receives 10 Mg data per second. An average IP package of data is about 1100 bytes. Thus, the average number of packages of data in a system is about 10000 packages per second.

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Received February 2010.