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SIMPLIFIED DESIGN OF LOW-PASS, LINEAR PARAMETER-VARYING, FINITE IMPULSE RESPONSE FILTERS

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Abstract. The main aim of the paper is to develop simplified tools and methods for design and analysis of linear parameter-varying (LPV) finite impulse response filters (FIR). FIR filters with constant coefficients have comprehensive theoretical foundations and design methods with the main advantages: good linearity of phase diagram, guaranteed stability, simple practical implementation. Although filters with constant coefficients guarantee particular properties in frequency domain, i.e. noise damping, they also increase rise time for rapid signal changes. In order to avoid such blurring effects a simplified design method for low-pass LPV FIR filters is developed. To synthesize the filter, two cut-off frequencies are needed accompanied with given filter order, shape tuning function and threshold detection condition for sequential operation. Quantitatively assess the filter quality and properties of the tuning functions are analyzed using both time and frequency dependent criteria. In the first case, difference Euclidean norm is used, while the frequency approach for filter analysis takes advantage of SVD-DFT transformation of linear time-varying discrete-time system, as previously defined by the author, employing singular value decomposition, discrete Fourier transformation and power spectral density properties.

Keywords: digital filters, finite impulse response filters, linear parameter varying systems, discrete-time systems, time-varying systems, non-stationary systems.

1. Introduction

Digital filters are one of fundamental tools for signal processing. The design and analysis methodology for linear time-invariant filters or filters with constant coefficients is currently well known [1] and can be easily applied by using a wide spectrum of software applications (e.g. Matlab Signal Processing Toolbox). In the simplest case, the design of a low pass filter can be simplified to finding only two parameters: the cut-off frequency and the filter order. However, classical time-invariant filters are an efficient tool for stationary processes or in the case when spectra of signal and noise are disjointed, as shown in Figure 1.

In many applications the signal and noise spectra are jointed and the cut-off frequency is reached as some compromise between the accepted level of noise and the accepted level of information loss in the signal. When the signal is non-stationary, e.g. the spectrum of the signal varies in time, the increase rise time for signal steps changes in classical stationary filters. It may be unexpected in some applications such as filtering signal from sensors, image processing etc. Such effects are caused by neglecting that the frequency spectra of the signal may vary with time.

Figure 2 shows signal spectra components, where S_{TI} denotes constant band of the signal, S_{TV} denotes the time-varying band of the signal, which appear only in selected time moments, whereas N denotes the noise band. In the simplest case, such a signal should be characterized by at least two limit frequencies: ω_d for the constant band and ω_m for the time-varying band.

Rise time for rapid signal changes may be improved by dynamically widening the filter band when the step change is detected. In such a case, the noise component is less important. In order to improve noise damping for small signal changes, the pass-band of the filter should be contracted.

This approach gives better results than a classical constant coefficient filter because specificity of the input signal is acknowledged and additionally group delay can correct what is equivalent to linearity of the phase diagram of the filter [2]. Most often the linear time-varying (LTV) filter (equivalently called linear parameter varying (LPV) filter) operates sequentially [2].

The main aim of the paper is to propose a simplified methodology for analysis and design of LPV finite impulse response (FIR) low-pass filters. In this paper, we propose two design methods for tuning parameters of the filter for two given cut-off frequencies and filter orders. Quantitatively assess the filter quality and properties of the tuning functions are analyzed using both time and frequency dependent criteria. In the first case, difference Euclidean norm is used, while the frequency approach for filter analysis takes advantage of SVD-DFT transformation of linear time-varying discrete-time system, as defined previously by the author [12], by employing singular value decomposition, discrete Fourier transformation and power spectral density properties.

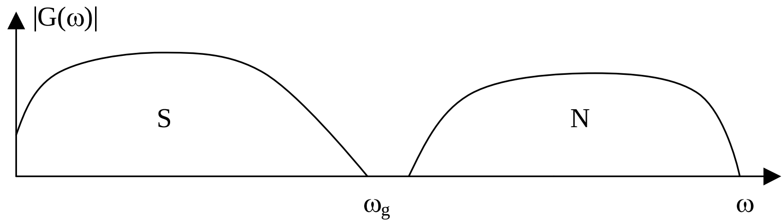

Figure 1. Frequency spectra of stationary signal with noise; S - signal, N – noise

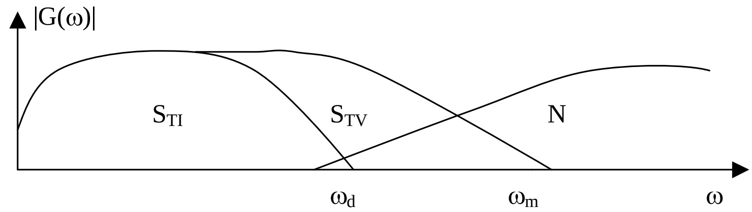

Figure 2. Frequency spectra of non-stationary signal with noise; S_{TI} – signal, stationary part, S_{TV} – signal, non-stationary part, N – noise

2. Filter Description

Usually, the parameter varying the FIR filter can be described by the following time-varying difference equation:

$$\begin{aligned} y(k) &= b_0(k)v(k) + b_1(k)v(k-1) + \\ &\ldots + b_n(k)v(k-n) \end{aligned}. \tag{1}$$

Alternatively, the description can be converted into a more general equation following operator description [13, 14]:

$$\hat{\mathbf{y}} = \hat{\mathbf{T}}\hat{\mathbf{v}}, \tag{2}$$

where $\hat{\mathbf{v}} = \begin{bmatrix} v(0) & \cdots & v(N-1) \end{bmatrix}^T$, $\hat{\mathbf{y}} = \begin{bmatrix} y(0) & \cdots & y(N-1) \end{bmatrix}^T$ are column vectors with filter input and output signals, respectively. Under assumption that the system is analyzed on a finite horizon N, the input-output operator $\hat{\mathbf{T}}$ is a compact, Hilbert-Schmidt operator from l_2 into l_2 and actually maps bounded signals $v(k) \in \mathcal{V} = l_2[0, N]$ into signals $y(k) \in \mathcal{Y}$.

The input-output operator $\hat{\mathbf{T}}$ of the system may be defined using a set of impulse responses of a time-varying system taken at different times in the following form:

$$\hat{\mathbf{T}} = \begin{bmatrix} h(0,0) & 0 & \cdots & 0 & 0 \\ h(1,0) & h(1,1) & \cdots & \vdots & \vdots \\ h(2,0) & h(2,1) & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & h(N-2,N-2) & 0 \\ h(N-1,0) & \cdots & \cdots & h(N-1,N-2) & h(N-1,N-1) \end{bmatrix}, \tag{3}$$

where $h(k_1, k_0)$ is the response of the system to the Kronecker delta $\delta(k-k_0)$ at time k_1 (after k_1-k_0 samples).

3. LPV FIR filter analysis and design

Although design of classical FIR filters can be simplified to the choice of two parameters (cut-off frequency, order), the design of LPV FIR filters requires at least a choice of three parameters (two cut-off frequencies and order). Sequentially operated LPV filters are each time-initiated by a particular event, e.g. by detection of step input signal. The threshold must be determined individually from the input signal analysis. LPV FIR filters can be designed using the following 3-stage process:

1. Select filter order R and cut-off frequencies: ω_m and ω_d, where ω_d is selected for steady-state (small signal changes) and ω_m is selected for signal switching (large signal changes).
2. Select a proper tuning function of the cut-off frequency of the filter.
3. Select the event detection threshold for sequentially operating filter.

Stages 1 and 3 are inseparably connected with the form of input signal. Selection of appropriate parameters can be made in a similar way as for classical filters. The most common topic in some case independent on the input signal is Stage 2 – selection of the tuning function. Thus, we focus further on Stage 2, including also time domain evaluation using a shape deformation coefficient and frequency analysis described in details e.g. in [12].

The cut-off frequency of the filter can be tuned in many ways. In this paper, we propose two tuning functions, defined either on the basis of function $\sin^n$ (SINN) or fundamental filter finite impulse response (FFIR).

3.1. Tuning function SINN

Function SINN define tuning process of the cut-off frequency ω_g of the time-varying filter on the basis of function $\sin^n$ in the following way:

$$\omega_g(k)=\begin{cases}\omega_d+(\omega_m-\omega_d)\sin^c\left(\pi\frac{k-k_0}{R}\right) & k_0\le k\le R+k_0\\ \omega_d & \text{otherwise}\end{cases}, \quad (4)$$

where ω_d is lower cut-off low-pass filter frequency (dominant noise), ω_m is upper cut-off low-pass filter frequency (widen pass-band in order to fasten rise-time), R is filter order, k_0 is the beginning time of the coefficient tuning corresponding to an event in the input signal, k is the discrete time and c is the tune shape coefficient. The above relationship allows the tune shape to be formed due to parameter c. It may also be the main disadvantage of the function for some application, as an appropriate c must be selected separately for each filter, especially for different R, ω_d and ω_g.

3.2. Tuning function FFIR

Selecting an additional parameter c can be avoided by introducing tuning function based on fundamental impulse response, i.e. impulse response of filter with cut-off frequency ω_d. The shape of the central non-negative part of the response is similar to the SINN function with adequate c. Mathematically the FFIR function can be described by the following:

$$\omega_g(k)=\omega_d+(\omega_m-\omega_d)\frac{b^0_{k-k_0}}{\max\limits_{0\le k\le n} b^0_k}, \quad (5)$$

where b_k are positive central coefficients of FIR filter with frequency ω_d. Central coefficients are defined as the following:

$$b^0_k=\begin{cases}b_k & \left\{k: |0.5n-k|\le k_1, \underset{\Delta_k\le k_1}{\forall} b_{0.5n\pm\Delta_k}>0\right\}\\ 0 & \text{otherwise}\end{cases}. \quad (6)$$

The above relationship can be used both for even order filters $\{0.5n, k, k_0, \Delta_k \in \mathcal{N}\}$ and for odd order filters $\{k\in\mathcal{N}, 0.5\pm\Delta_k\in\mathcal{N}, k_0+0.5\in\mathcal{N}\}$. Graphical interpretation of central coefficients with respect to impulse response of filters with frequency ω_d is depicted in Figure 3. The main advantage of the FFIR function is the fact that there is no longer needed any additional parameter. Thus, the design procedure can be simplified to the selection of only 3 parameters: filter order R, cut-off frequencies ω_d, ω_g and additionally for sequentially operated filters event detection.

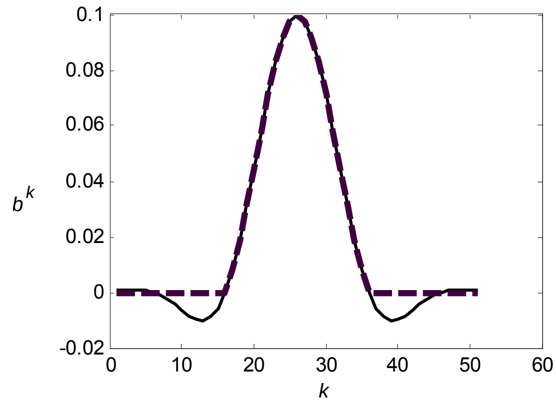

Figure 3. Central coefficients (dashed line) for FFIR tuning function originated from filter impulse response (solid line), R=50, ω_g =0.1

3.3. Obtaining Time-Varying filter coefficients

Coefficients $\mathbf{b}(k)=[b_0(k), b_1(k), \ldots, b_n(k)]$ of filters (4), (5) are computed using standard and available procedures (e.g. Matlab function b = fir1(n, Wn)). The procedure is directly adopted from classical time-invariant filters synthesis using Hamming window for impulse response of ideal filter [1]. Coefficients are computed from the following equation:

$$b_i(k)=w(i)h_k(i) \quad 1\le i\le R, \quad (7)$$

where $w(i)$ denotes Hamming window and $h_k(i)$ impulse response of ideal filter obtained by inverse Fourier transform of ideal low-pass filter frequency diagram with cut-off frequency $\omega_g(k)$.

3.4. Shape deformation coefficient

To quantitatively assess the properties of the filter and quality of the filtered signal, we introduce shape deformation coefficient (SDC). SDC is defined as a Euclidean vector norm of difference between filter output and reference filter output. Mathematically it can be written as follows:

$$SDC=\sqrt{\sum_{k=0}^{N-1}\left(y(k)-y_r(k)\right)^2}, \quad (8)$$

where $y(k)$ is output response in time domain of analyzed filter for given input signal, e.g. step signal, and $y_r(k)$ is output response of reference filter for the same signal. We assume two reference filters: filters with a cut-off frequency ω_m or all-pass filters both with the same order as the analyzed filter. There are also two possibilities to choose input signal of analyzed filter: signal $y(k)$ may be either a response of input signal with or without noise. In the second case, it is possible to evaluate the impact of the input noise to the output of the filter.

4. Numerical examples

In this section, we design and analyze LPV filters using proposed methodology for given R=50, ω_d =0.1, ω_m =0.6. We compare performance between FFIR and SINN functions with different values of parameter c, different noise levels and additional time shift.

4.1. Time and frequency analysis of LPV filters tuned with SINN function

Additional parameters assumed for the example are as follows: time shift k_0 = 1, standard deviation of Gaussian noise σ = 0.1, simulation horizon N = 100 steps, SINN function coefficients: c = 30 and c = 10. Computations are carried out for the following four filters:

1 – time-invariant filter with cut-off frequency $\omega_g = \omega_d$,

2 – parameter-varying filter tuned with SINN function with c = 30,

3 – parameter-varying filter tuned with SINN function with c = 10,

4 – time-invariant filter with cut-off frequency $\omega_g = \omega_m$.

Step responses for all filters are depicted in Figure 4, whereas tuning functions are shown in Figure 5. Input step signal contains additive input white noise with standard deviation σ = 0.1 and zero mean value. Time response is delayed by R/2 due to general properties of FIR filters.

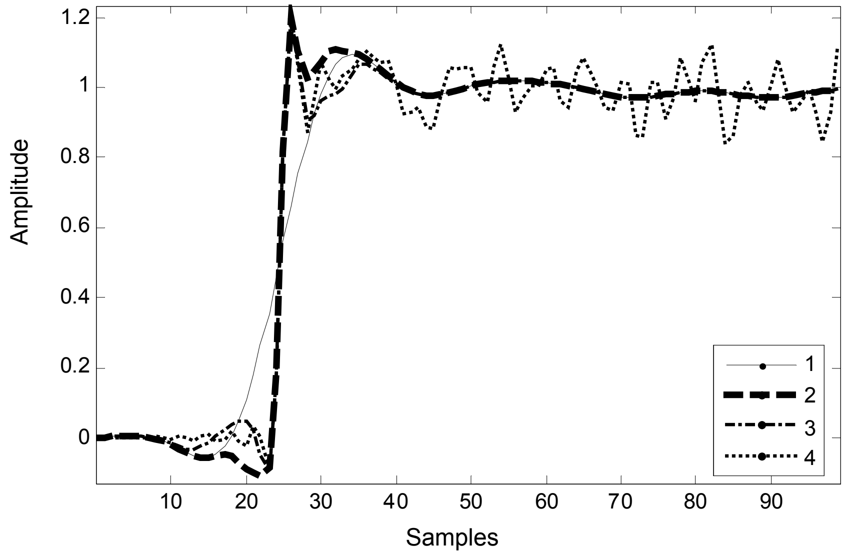

Figure 4. Step responses for time-invariant filters (1 – ω_g = 0.1, 4 – ω_g = 0.6) and for parameter-varying filters (2 – SINN c = 30,3 – SINN c = 10), R = 50, N = 100, k_0=1

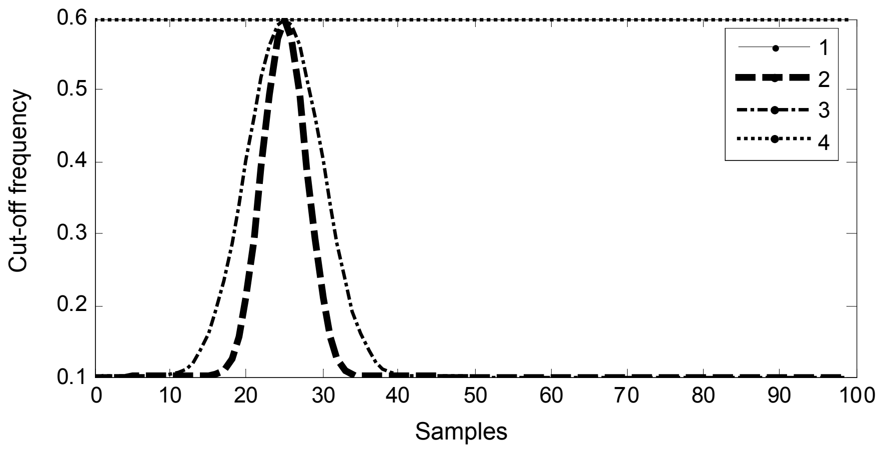

Figure 5. Cut-off frequency vs. discrete time (1 – ω_g =0.1, 2 – SINN c=30,3 – SINN c=10, 4 – ω_g =0.6), R=50, N=100, k_0=1

Approximated Bode diagrams depicted in Figure 6 are computed using system operator based on the set of time-shifted impulse responses of the system described in Section 2. The diagrams were introduced in [12]. The key steps needed to obtain approximated Bode diagrams are given in Table 1, while detailed algorithm can be found in [12]. The method requires the pre-processing of a system operator using singular value decomposition (SVD). Such decomposition presents a generalization of the classic SVD of matrices [16] (operators defined for discrete-time systems over a finite time horizon are finite dimensional). The magnitude diagram may be interpreted as the worst-case amplification (the maximal value), while the phase diagram represents only an approximated value (averaged or expected phase shift for given frequency). Selected examples of approximate Bode diagrams for first order systems can be found in [15].

Table 1. Key steps in SVD-DFT Bode diagrams approximation

	Approximated Bode diagrams
Input	Input-output system operator $\hat{\mathbf{T}}$ lower triangular
Pre-processing	Singular Value Decomposition $\hat{\mathbf{T}} = \mathbf{USV}^T$
Transformation	Column DFT of matrices **U** and **V**
Post processing	Square average using weights from **S** matrix
Result	Two 2D real diagrams: magnitude and phase vs. frequency. For LTI systems equivalent to classical Bode diagrams.

As it is shown in Figure 6, the cut-off frequencies (-3dB) of filters 1–3 are almost the same, nevertheless parameter-varying filters 2, 3 have weaker damping for high frequencies. The weaker damping follows from temporally raised cut-off frequency. The damping for parameter-varying filter should be approximately between diagrams 1 and 4. The longer or more often tuned, the weaker damping of the filter for high frequencies.

Phase diagrams are dependent on the starting time of the analysis. For example, for time shift k_0=25, linear range of the phase for filters 2, 3 ends at $\omega \approx 0.64$ but is still 3-times longer compared to filter 1. Similar conclusions can be made by comparing to phase diagrams from Figure 9 with the next example where phase diagram is also dependent on time shift (k_0 = 30). In practice it is sufficient for the phase to be linear for rapid signal changes, otherwise phase is less important.

4.2. Comparative analysis of filters tuned with FFIR and SINN functions

Additional parameters assumed for the example are the following: time shift k_0 = 30, standard deviation of Gaussian noise σ = 0.2, simulation horizon N = 110 steps. Tuning functions are FFIR and SINN with c = 50.

Computations are carried out for the following four filters:

1 – time-invariant filter with cut-off frequency $\omega_g = \omega_d$,

2 – parameter-varying filter tuned with FFIR,

3 – parameter-varying filter tuned with SINN function with c = 50,

4 – time-invariant filter with cut-off frequency $\omega_g = \omega_m$.

Step responses for all four filters are depicted in Figure 7. Tuning functions for filter cut-off frequencies are plotted in Figure 8. Step input signal contains additive input white noise with standard deviation σ =0.2 and zero mean value.

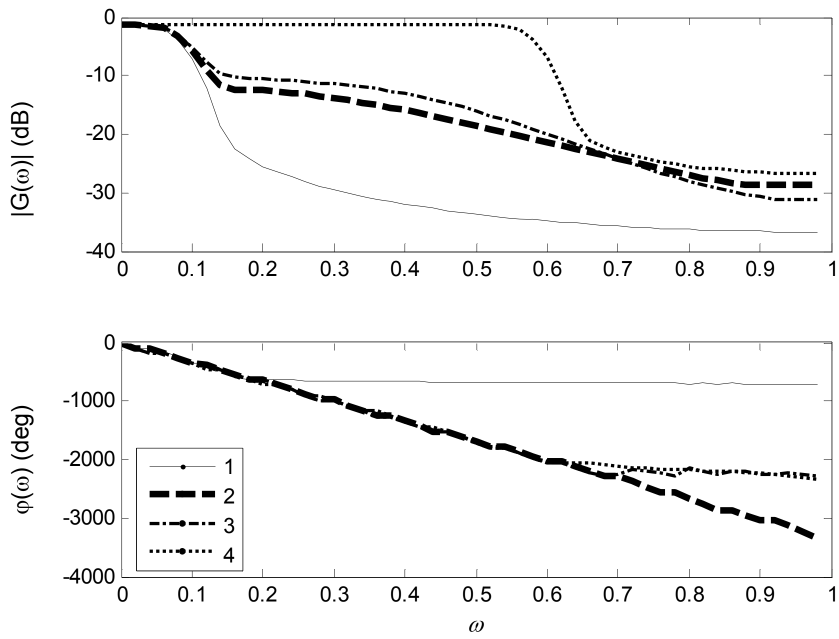

Figure 6. Approximate Bode diagrams for time-invariant filters (1 – ω_g = 0.1, 4 – ω_g = 0.6) and for parameter-varying filters (2 – SINN c =3 0,3 – SINN c = 10), R = 50, N = 100, k_0 = 1

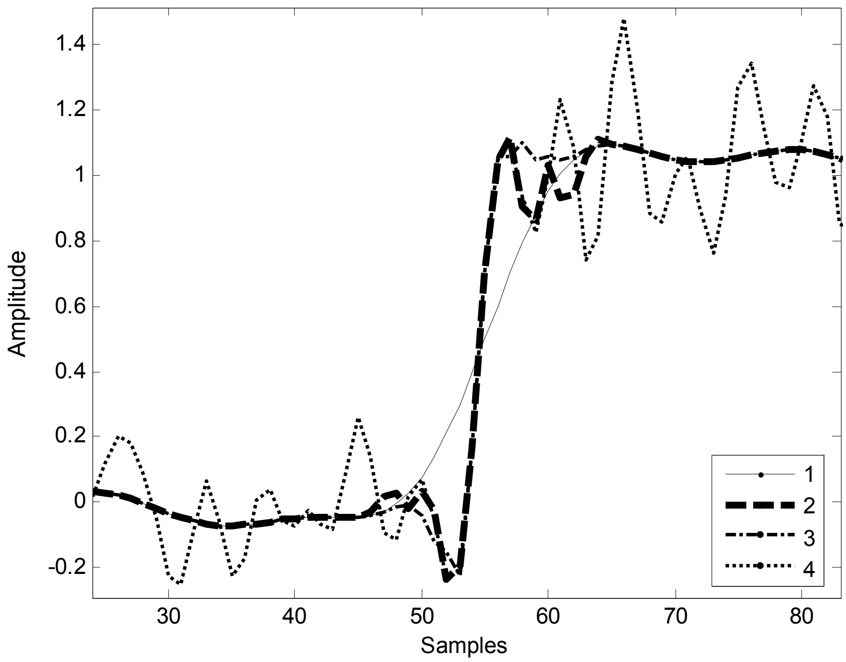

Figure 7. Step responses for time-invariant filters (1 – ω_g = 0.1, 4 – ω_g = 0.6) and for parameter-varying filters (2 – FFIR,3 – SINN c = 50), R = 50, N = 110, k_0 = 30

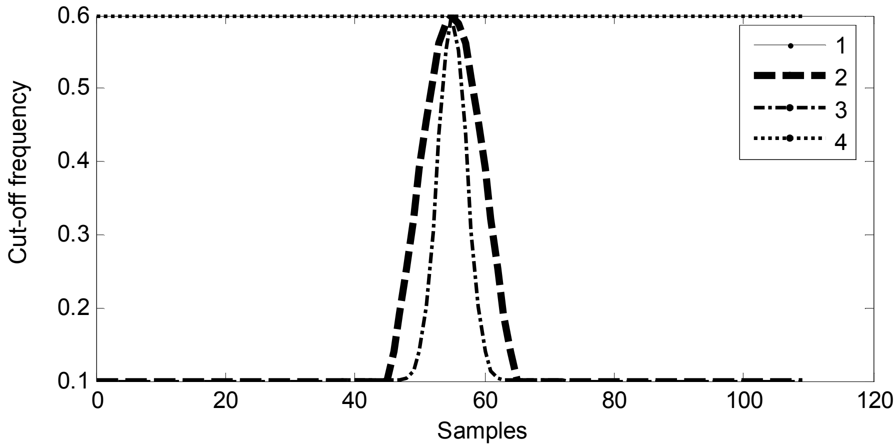

Figure 8. Cut-off frequency vs. discrete time (1 – ω_g = 0.1, 2 – FFIR, 3 – SINN c = 50, 4 – ω_g = 0.6), R = 50, N = 110, k_0 = 30

As was mentioned earlier, the quality of the disturbance filtration is a certain compromise between the distortions due to filtration of useful signal without noise and the disturbances level after filtration due to noise in the input signal. Comparative time domain evaluation of filtration can be done due to the shape deformation coefficient defined in Section 3.4. Results of the analysis are shown in Table 2. Coefficients SDC_1 and SDC_3 are evaluated for reference filters equal to filter 4 while SDC_2 and SDC_4 are referenced by all-pass filter. Coefficients SDC_1, SDC_2 are evaluated for input signal without noise whereas SDC_3, SDC_4 are computed in presence of additive input noise with σ =0.2.

Results collected in Table 2 show that:

a) for similar width of tuning functions FFIR and SINN c=10 values of corresponding coefficients are similar;

b) for higher c = 50, 100 (narrower SINN function), all coefficients increase with simultaneous higher average damping of the filter – larger distortions, larger noise damping;

c) for lower c = 1 (wider SINN function) noised coefficients increase, while coefficients without noise (SDC_1, SDC_2) decrease – smaller distortions, smaller noise damping.

Approximated Bode diagrams represent properties of the system for all simulation horizons. Interpretation of the results is an easy task and can be made by analogy to classical time-invariant systems. Additionally noised system operators can be also filtered thanks to the noise-robustness properties of singular value decomposition.

Table 2. Shape deformation coefficients for different tuning functions of the filter

Filter	SDC_1	SDC_2	SDC_3	SDC_4
LTI ω_d	0.88	0.95	0.97	1.03
FFIR	0.21	0.37	**0.65**	**0.70**
SINN c = 1	0.07	0.33	1.11	1.17
SINN c = 10	0.17	0.35	**0.65**	**0.71**
SINN c = 50	0.28	0.39	0.70	0.73
SINN c = 100	0.33	0.42	0.83	0.88
LTI ω_m	0	0.33	1.07	1.11

5. Conclusion

The main difficulty for parameter varying filter design is to choose an appropriate tuning function. We analyzed two different tuning functions SINN and FFIR. Although, for similar widths, both SINN and FFIR functions have similar properties, it is shown that FFIR function has the best time response (lower value of shape deformation coefficient) for noised signal (Table 2 – SDC_3, SDC_4 values in bold face) referenced by all-pass filter. SINN function with a well-chosen parameter c has comparable values of noised SDC (see e.g. Table 2 – SDC_3, SDC_4 : FFIR and SINN c = 10). Higher values of c lead to larger deformation of step response but simultaneously improve the damping factor for higher frequencies. Lower values of c are generally undesirable because they decrease damping factor for high frequency noise. The proposed methodology can be applied to simplified design of LPV filter in a similar way as for classical time-invariant systems. Described previously method for approximated Bode diagrams applied to digital LPV filters allows to plot simplified frequency spectra for time-varying systems.

Acknowledgments

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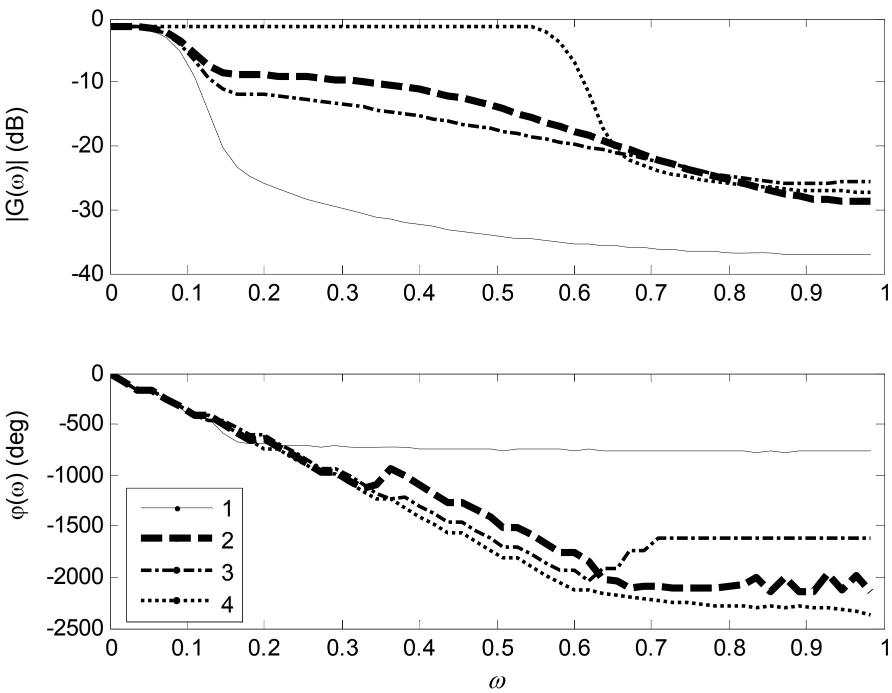

Figure 9. Approximate Bode diagrams for time-invariant filters (1 – ω_g = 0.1, 4 – ω_g = 0.6) and for parameter-varying filters (2 – FFIR, 3 – SINN c = 50), R = 50, N = 110, k_0 = 30

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