ON APPLYING SPARSE AND UNCERTAIN INFORMATION TO ESTIMATING THE PROBABILITY OF FAILURE DUE TO RARE ABNORMAL SITUATIONS

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Abstract. Estimating the probability of failure due to a rare and abnormal situation may face the need to deal with information which is incomplete and involves uncertainties. Two sources of information are applied to this estimating: a small-size statistical sample and a fragility function. This function is used to express aleatory and epistemic uncertainties related to the potential failure. The failure probability is estimated by carrying out Bayesian inference. Bayesian prior and posterior distributions are applied to express the epistemic uncertainty in the failure probability. The central problem of probability estimating is formulated as Bayesian updating with imprecise data. Such data are represented by a set of continuous epistemic probability distributions of fragility function values related to elements of the small-size sample. The Bayesian updating with the set of continuous distributions is carried out by discretising these distributions. The discretisation yields a new sample used for updating. This sample consists of fragility function values which have equal epistemic weights. Several aspects of numerical implementation of the discretisation and subsequent updating are discussed and illustrated by two examples.

Keywords: failure probability, fragility, small-size sample, imprecise data, epistemic uncertainty, Bayesian updating.

1. Introduction

Abnormal situations occurring during exploitation of technical objects are among the main reasons for failures of these objects. As a rule, an abnormal situation is a highly random event of short duration which can be caused (triggered off) by component failure, human error, man-made accident, extreme natural phenomenon. Abnormal situations are a natural subject of the quantitative risk assessment (QRA) [1-5]. QRA applies Bayesian reasoning to a systematic quantification of risk posed by these situations [6, 7]. A probability of failure of a technical object subjected to an abnormal situation can be component of such risk [8-10].

One of the main approaches to estimating the probability of failure due to an abnormal situation is a decomposition of the problem into two subproblems: (i) predicting characteristics of abnormal situation and (ii) modelling the fragility of object subjected to this situation. The fragility is quantified in terms of conditional failure probabilities expressed as a fragility function. Such a decomposition is widely used, for instance, in the earthquake risk assessment [11-15] and extreme wind risk assessment [16, 17]. A solution of the subproblems (i) and (ii) may face the need to deal with sparse and uncertain information related to both abnormal situation and potential failure due to this situation.

Methods proposed for estimating probabilities using sparse and uncertain information are numerous. These methods are based either on fuzzy logic, probability theory, possibility theory, or evidence theory and their general purpose is modelling uncertainties [11, 13, 18-23]. In line with QRA, these uncertainties are divided into aleatory (irreducible) one and epistemic (reducible) one (e.g. [6, 7]). Although a comparative state-of-art review of the aforementioned methods is not available, one can state that some of them can be used for modelling uncertainty in the fragility to an abnormal situation. In case where this modelling is done in the Bayesian format, the failure probability can be estimated by carrying out Bayesian inference which takes the mathematical form of Bayesian updating with imprecise (fuzzy) data [5, 8, 9, 24]. Such data is generated in the course of estimation and represented by continuous epistemic probability distributions of fragility function values. These distributions express the epistemic uncertainty inherent in the fragility function.

A relatively small number of approaches were proposed to solve the problem of Bayesian updating with imprecise data [25-28]. In the case where the data uncertainty are expressed by epistemic probability distributions, the only practicable approaches seem to be either averaging out data uncertainties ("data averaging approach") or averaging conventional Bayesian posterior distributions ("posterior averaging approach") (see articles [25] and [28] and the references therein). By their nature, both approaches are heuristic ad hoc procedures.

The data averaging is a trivial procedure which replaces the epistemic distributions of data uncertainty by mean values of these distributions. Unfortunately, most of the information expressed by the epistemic distributions is lost due to such averaging (can not be further propagated). The posterior averaging is a more sophisticated approach. It is based on the use of discrete distributions quantifying epistemic uncertainty in individual data points. This data must have a relatively simple form of a single uncertain datum, for instance, number of failures [25, 28]. The latter approach is not directly applicable to the case where the data uncertainty is modelled by continuous distributions. Such a case is considered in the present paper.

A discretisation of continuous distributions of data uncertainty could be a help in applying the posterior averaging approach. This paper shows that a special kind of discretisation allows to dispense with the posterior averaging and to create a set of data which can directly enter into Bayes theorem through a likelihood function. This discretisation is applicable to the aforementioned epistemic distributions of fragility function values. A Bayesian updating with the data generated by discretising these distributions will yield a posterior distribution expressing the epistemic uncertainty in the failure probability under estimation.

2. Problem and background information

Let the random event $\mathcal{F}$ denote a potential failure of a technical object due to an abnormal situation which is represented by the random event $\mathcal{AS}$. The conditional probability of $\mathcal{F}$ can be expressed in the form of a mean value [5, 8, 9, 24]:

$$\mu \equiv P(\mathcal{F}|\mathcal{AS}) = \int_{\text{all } y} P(\mathcal{F}|\, \boldsymbol{y})\, \mathrm{d}F_{\boldsymbol{Y}}(\boldsymbol{y}) = E_{\boldsymbol{Y}}(P(\mathcal{F}|\, \boldsymbol{Y}))\,, \qquad (1)$$

where $\boldsymbol{Y}$ is the random vector of the characteristics which represent the abnormal situation; $\boldsymbol{y}$ and $F_{\boldsymbol{Y}}(\boldsymbol{y})$ are the value of $\boldsymbol{Y}$ and its joint distribution function (d.f.), respectively; $P(\mathcal{F}|\,\boldsymbol{y})$ is the conditional probability of $\mathcal{F}$ given $\boldsymbol{y}$; $E_{\boldsymbol{Y}}(\cdot)$ denotes the mean value with respect to $\boldsymbol{Y}$; and $P(\mathcal{F}|\boldsymbol{Y})$ denotes a function of the random vector $\boldsymbol{Y}$. $P(\mathcal{F}|\,\boldsymbol{y})$ relates the particular value $\boldsymbol{y}$ to the probability of $\mathcal{F}$ and is called the fragility function (f.f.). Its arguments $\boldsymbol{y}$ are called the demand variables (e.g. [11, 13]).

In terms of QRA, the function $P(\mathcal{F}|\,\boldsymbol{y})$ expresses the aleatory uncertainty in occurrence of $\mathcal{F}$ given an abnormal situation with characteristics $\boldsymbol{y}$. However, values of $P(\mathcal{F}|\,\boldsymbol{y})$ can be uncertain in the epistemic sense. Several different approaches were proposed to model the epistemic uncertainty in $P(\mathcal{F}|\,\boldsymbol{y})$ [11-13, 16, 29]. A systematic review of these approaches is not available at present. However, the most consistent approach seems to be developing $P(\mathcal{F}|\,\boldsymbol{y})$ by means of Bayesian parameter estimation [11, 13]. The epistemic uncertainty in $P(\mathcal{F}|\,\boldsymbol{y})$ is expressed by means of the Bayesian limit state function $g(\boldsymbol{z}, \boldsymbol{y}\,|\,\boldsymbol{\theta})$. In this function, $\boldsymbol{z}$ is the vector describing the technical object exposed to an abnormal situation and $\boldsymbol{\theta}$ denotes the vector of model parameters. With a fixed (crisp) $\boldsymbol{\theta}$, $P(\mathcal{F}|\,\boldsymbol{y})$ expresses aleatory uncertainty only and is defined as

$$F(\boldsymbol{y}) \equiv P(\mathcal{F}|\,\boldsymbol{y}) = P(g(\boldsymbol{Z}, \boldsymbol{y}\,|\,\boldsymbol{\theta}) \le 0), \qquad (2)$$

where $\boldsymbol{Z}$ is the random vector quantifying the aleatory uncertainty in $\boldsymbol{z}$.

A possible epistemic uncertainty in $\boldsymbol{\theta}$ is modelled by a random vector $\boldsymbol{\Theta}$ with a joint probability density function (p.d.f.) $\pi(\boldsymbol{\theta})$ and joint d.f. $F_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. In the Bayesian framework, the p.d.f. $\pi(\boldsymbol{\theta})$ is treated as a prior distribution which can be updated by means of the standard Bayesian procedure [11, 13]. With the random $\boldsymbol{\Theta}$, the f.f. $P(\mathcal{F}|\,\boldsymbol{y})$ for a given $\boldsymbol{y}$ becomes an epistemic random variable (r.v.). Such an f.f. will quantify both aleatory and epistemic uncertainty:

$$F(\boldsymbol{y}\,|\,\boldsymbol{\Theta}) \equiv P(\mathcal{F}\,|\,\boldsymbol{y}, \boldsymbol{\Theta}) = P(g(\boldsymbol{Z}, \boldsymbol{y}\,|\,\boldsymbol{\Theta}) \le 0). \qquad (3)$$

For brevity sake, the functions $F(\boldsymbol{y})$ and $F(\boldsymbol{y}\,|\,\boldsymbol{\Theta})$ will be called the aleatory f.f. and the epistemic f.f., respectively. The following consideration seeks to answer the question, how to estimate $P(\mathcal{F}|\mathcal{AS})$ by applying two sources of information about the abnormal situation under analysis: (a) the aleatory and epistemic f.f.s $F(\boldsymbol{y})$ and $F(\boldsymbol{y}\,|\,\boldsymbol{\Theta})$; (b) a small-size statistical sample $\mathbf{y}$ consisting of experimental observations of $\boldsymbol{y}$:

$$\mathbf{y} = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_j, \ldots, \boldsymbol{y}_n\}\,, \qquad (4)$$

where $\boldsymbol{y}_j$ is the value of $\boldsymbol{y}$ recorded in the jth experiment. The case is considered where the size n of $\mathbf{y}$ is too small to fit the d.f. $F_{\boldsymbol{Y}}(\boldsymbol{y})$ in the standard statistical way. The case of the small n is considered to be realistic one because experiments imitating an abnormal situation can be too expensive to obtain a large-size $\mathbf{y}$.

3. Estimating the failure probability with the aleatory fragility function

3.1. Developing the prior density

The mean value μ defined by Eq (1) is amenable to Bayesian inference. The prior $\pi(\mu)$ of μ can be specified by utilizing knowledge about the abnormal situation under study [9, 24]. Such knowledge, more or less relevant to the situation, can often be represented by the mathematical model

$$\boldsymbol{y} = \boldsymbol{\nu}(\boldsymbol{x} \mid \boldsymbol{\xi}), \quad (5)$$

where $\boldsymbol{x}$ is the vector which represents information allowing to predict the characteristics $\boldsymbol{y}$; $\boldsymbol{\xi}$ is the vector of parameters of the vector function $\boldsymbol{\nu}(\cdot)$ which are uncertain in the epistemic sense. Information represented by $\boldsymbol{x}$ may be uncertain in the aleatory sense and this uncertainty can be modelled by a random vector $\boldsymbol{X}$ with an aleatory d.f. $F_{\boldsymbol{X}}(\boldsymbol{x})$. Epistemic uncertainties related to $\boldsymbol{\xi}$ can be expressed by introducing a random vector $\boldsymbol{\Xi}$ with a d.f. $F_{\boldsymbol{\Xi}}(\boldsymbol{\xi})$.

Replacing $\boldsymbol{Y}$ in the function $P(\mathcal{F}|\boldsymbol{Y})$ by the random function $\boldsymbol{\nu}(\boldsymbol{X} \mid \boldsymbol{\Xi})$ and averaging out the aleatory uncertainty expressed by $\boldsymbol{X}$ yield the epistemic r.v.

$$M = E_{\boldsymbol{X}}(F(\boldsymbol{\nu}(\boldsymbol{X} \mid \boldsymbol{\Xi}))) = \int_{\text{all } \boldsymbol{x}} F(\boldsymbol{\nu}(\boldsymbol{X} \mid \boldsymbol{\Xi}))\mathrm{d}F_{\boldsymbol{X}}(\boldsymbol{x}). \quad (6)$$

A value of M is the failure probability at given $\boldsymbol{\xi}$. A density of M can be used as the prior $\pi(\mu)$ quantifying the epistemic uncertainty in $P(\mathcal{F}|\mathcal{AS})$ [9, 24].

3.2. New data

The potential abnormal situation may be unique by a large margin and so may not fit fully in the prior knowledge expressed by the model $\boldsymbol{\nu}(\cdot)$. The source of the partial irrelevance may lie in structure of $\boldsymbol{\nu}(\cdot)$ and/or data used to fit the d.f. $F_{\boldsymbol{X}}(\boldsymbol{x})$ and estimate the parameters $\boldsymbol{\xi}$.

The new data necessary for estimating μ can be derived from the sample $\mathbf{y}$. This sample can be used for estimating μ if it possesses the property of statistical representativeness and is relevant to the abnormal situation under analysis.

Given the sample $\mathbf{y}$ and the aleatory f.f. $F(\boldsymbol{y})$, one can simplify estimating μ by introducing a fictitious sample

$$\boldsymbol{p} = \{p_1, p_2, \ldots, p_j, \ldots, p_n\}. \quad (7)$$

The element p_j of $\boldsymbol{p}$ is equal to $F(\boldsymbol{y}_j)$. The introduction of $\boldsymbol{p}$ allows to simplify the estimation problem by switching from a multi-dimensional analysis to a one-dimensional case.

3.3. Updating procedure

The usual Bayesian posterior $\pi(\mu \mid \text{data})$ is proportional to the product $\pi(\mu)\times L(\text{data} \mid \mu)$, where $L(\text{data} \mid \mu)$ is the likelihood function and "data" is represented by the sample $\boldsymbol{p}$. The posterior $\pi(\mu \mid \text{data})$ can be replaced by an estimated one [8, 9, 24]:

$$\hat{\pi}(\mu \mid \text{data}) \propto \pi(\mu)\hat{L}_B(\text{data} \mid \mu), \quad (8)$$

where $\hat{L}_B(\text{data} \mid \mu)$ is an estimate of $L(\text{data} \mid \mu)$ based on bootstrap estimation of the density of the pivotal quantity $\hat{\mu}_n - \tilde{M}$, where $\hat{\mu}_n$ is the mean value of the sample $\boldsymbol{p}$.

The estimate $\hat{L}_B(\text{data} \mid \mu)$ is calculated by the following expression [30]:

$$\hat{L}_B(\hat{\mu}_n \mid \mu) = \frac{1}{B\,w}\sum_{b=1}^{B}\kappa\left(\frac{2\hat{\mu}_n - \mu - \hat{\mu}'_{nb}}{w}\right), \quad (9)$$

where B is the number of random bootstrap samples of the size n generated from the empirical d.f. $\hat{F}_n$ of $\boldsymbol{p}$; $\hat{\mu}'_{nb}$ is the mean value of the bth bootstrap sample; $\kappa(\cdot)$ is the kernel function (e.g., density of standard normal distribution); w is a bandwidth (window width, smoothing parameter).

The resulting $\hat{\pi}(\mu \mid \text{data})$ is obtained by

$$\hat{\pi}(\mu \mid \hat{\mu}_n) = C(\hat{\mu}_n)\pi(\mu)\hat{L}_B(\hat{\mu}_n \mid \mu), \quad (10)$$

where $C(\hat{\mu}_n)$ is the normalizing constant.

Computational implementation of the bootstrap-based updating procedure is relatively simple. The estimates $\hat{L}_B(\hat{\mu}_n \mid \mu)$ and $\hat{\pi}(\mu \mid \hat{\mu}_n)$ can be computed almost automatically (see, e.g., the book [31] for details).

3.4. First example: the use of aleatory fragility function

3.4.1. Prior knowledge

The failure probability $P(\mathcal{F}|\mathcal{AS})$ is to be estimated for an abnormal situation which can be caused by an accidental explosion within a 150×200 m^2 zone of a plant processing industrial explosives (Figure 1). The failure $\mathcal{F}$ consists in a loss of containment of a steel tank built outside the zone due to action of the blast wave generated by the explosion [32-34]. The prior knowledge is expressed by the model

$$y = \nu(\boldsymbol{x}|\xi) = \nu'(\xi\psi(x_1, x_2))$$
$$= \nu'\left(\xi\left(\frac{0.1\,x_1^{1/3}}{r(x_2, x_3)} + \frac{0.43x_1^{2/3}}{r^2(x_2, x_3)} + \frac{1.4x_1}{r^3(x_2, x_3)}\right)\right), \quad (11)$$

where y is the peak positive overpressure of the blast wave reflected by the tank; $r(x_1, x_2)$ is the standoff of the explosion (Figure 1); $\nu'(\cdot)$ is the deterministic function used to transform the incident peak overpressure into the reflected one [35]; ξ is the dimensionless factor used to adjust the standard trinitrotoluol model $\psi(x_1, x_2)$ to the explosive under analysis.

The aleatory uncertainty is related to arguments of $\boldsymbol{\nu}(\boldsymbol{x} \mid \xi)$ and expressed by the random vector $\boldsymbol{X} = (X_1, X_2, X_3)^{\mathrm{T}}$. Its components are the normally distributed mass of explosive, X_1~N(30 kg, 3 kg), and the uniformly distributed coordinates of explosion centre, X_2~U(0 m, 150 m) and X_3~U(0 m, 200 m). The epistemic uncertainty is introduced into the prior knowledge by assuming that the adjustment factor ξ is uncertain in the epistemic sense. This uncertainty is modelled by a lognormal r.v. $\Xi \sim$ L(0.17975, 0.11957) (with the mode of 1.17 and the coefficient of variation equal to 0.15).

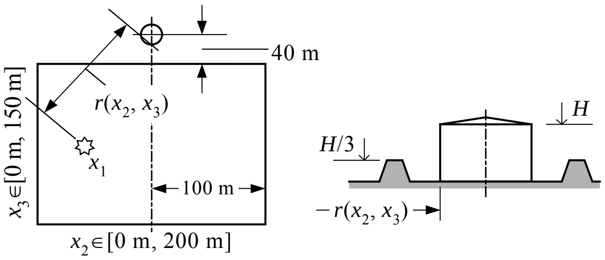

Figure 1. The site on which the abnormal situation may occur

The vulnerability of the tank to the explosion is expressed by the aleatory f.f. $F(y \mid \boldsymbol{\theta})$ represented by the normal d.f. with fixed parameters $\boldsymbol{\theta} = (\theta_1, \theta_2)^{\mathrm{T}}$ = (7 kPa, 0.7875 (kPa)^2)$^{\mathrm{T}}$.

3.4.2. Prior density of the failure probability

The prior density $\pi(\mu)$ can be specified by fitting it to the sample $\{\mu_1, \mu_2, \ldots, \mu_l, \ldots, \mu_{n_l}\}$. The sample element μ_l is an estimate of the mean value $E_{\boldsymbol{X}}(F(\boldsymbol{\nu}(\boldsymbol{X} \mid \xi_l) \mid \boldsymbol{\theta}))$ at the given value ξ_l of the epistemic r.v. Ξ (see Eq (6)). To generate the sample of μ_ls, the values ξ_l were sampled by means of a stochastic (Monte Carlo) simulation from L(0.17975, 0.11957).

The sample size n_l was chosen to be 1000. A lognormal p.d.f. $\pi(\mu \mid -2.71691, 0.519298)$ was fitted to the sample of μ_ls as the prior density expressing the epistemic uncertainty in $P(\mathcal{F} \mid \mathcal{AS})$ (Figure 2). The goodness of fit of the density $\pi(\mu)$ shown in Figure 2 is, strictly speaking, low. However, the ideal fit is not an end in itself. The density $\pi(\mu)$ merely quantifies the initial guess at $P(\mathcal{F} \mid \mathcal{AS})$. Therefore $\pi(\mu)$ can be subjective to some extent (not fit ideally the simulated sample of μ_ls).

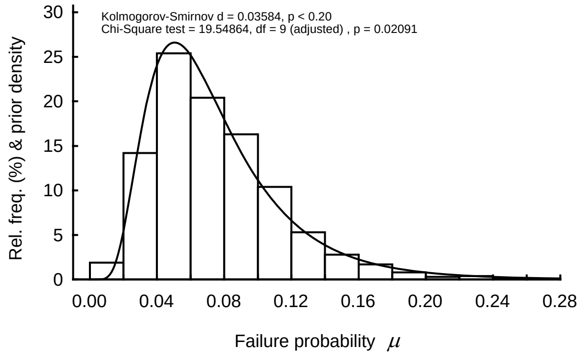

Figure 2. Histogram of the sample $\{\mu_1, \mu_2, \ldots, \mu_{1000}\}$ and the modified lognormal prior $\pi(\mu \mid -2.71691, 0.519298)\times 20\%$ fitted to this sample

3.4.3. New information used for updating

The model $\boldsymbol{\nu}(\boldsymbol{x} \mid \xi)$ is only partially relevant to the situation shown in Figure 1. It is valid for a distant free-field explosion on the ground which forms a horizontal plane. However, the tank is surrounded by a circular protective soil embankment. This will significantly influence the blast wave and reduce the reflected overpressure y. Thus the model $\boldsymbol{\nu}(\boldsymbol{x} \mid \xi)$ is sufficient only to specify the prior density $\pi(\mu)$. This should be updated using new data $\mathbf{y}$.

The new data $\mathbf{y}$ were obtained from a series of nine experiments which investigated the interaction of blast wave and circular embankment (n = 9). Elements of the sample $\mathbf{y}$ are given in Table 1. This sample was transformed into the sample of f.f. values, $\boldsymbol{p}$, by applying the aleatory f.f. $F(y \mid \boldsymbol{\theta})$.

Table 1. New data $\mathbf{y}$ (experimental records of the overpressure y_j) and corresponding sample of f.f. values, $\boldsymbol{p}$

j	Charge (kg)	Standoff (m)	y_j (kPa)	$p_j = F(y_j \mid \boldsymbol{\theta})$
Samples obtained in experiment*				Fictitious sample
1	27.0	117	3.767	1.3450089×10^{-4}
2	26.9	142	4.276	1.0697380×10^{-3}
3	28.2	132	4.160	6.8615251×10^{-4}
4	31.5	125	3.944	2.8665579×10^{-4}
5	29.3	92	4.916	9.4388105×10^{-3}
6	33.3	50	2.920	2.1316347×10^{-6}
7	30.0	119	4.791	6.4023419×10^{-3}
8	34.6	86	4.032	4.1149950×10^{-4}
9	33.0	39	2.294	5.6915293×10^{-8}

* Data obtained by the author of this paper

3.4.4. Posterior of failure probability

The number of bootstrap replications, B, necessary to generate the sample $\{\hat{\mu}'_{n1}, \hat{\mu}'_{n2}, \ldots, \hat{\mu}'_{nB}\}$ was taken to be equal to 1000. The choice of B was based on the rules of thumb suggested in the books [36, p. 52] and [37, p. 21]. The estimate of the likelihood function, $\hat{L}_{1000}(\hat{\mu}_9 \mid \mu)$, was obtained by applying the Gaussian kernel function $\kappa(.)$ (e.g. [37, p. 168]).

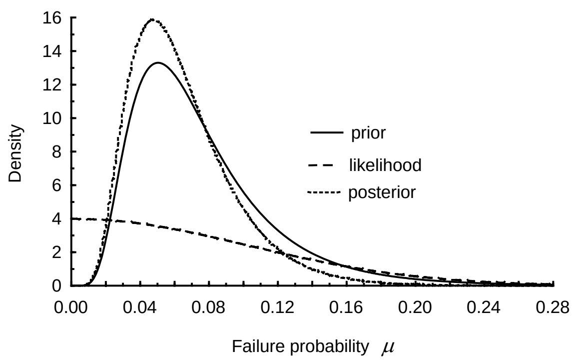

Figure 3. Prior density $\pi(\mu)$ and the likelihood function estimate $\hat{L}_{1000}(\hat{\mu}_9 \mid \mu)$ and posterior density estimate $\hat{\pi}(\mu \mid \hat{\mu}_9)$

The approximation of the posterior density $\hat{\pi}(\mu \mid \hat{\mu}_9)$ was computed at the bandwidth $w = 0.1$. This value was chosen using the rule $w \propto B^{-1/3}$ proposed by Davison and Henley [37, p. 227]. The approximation $\hat{\pi}(\mu \mid \hat{\mu}_9)$ was obtained by a numerical calculation. The normalizing constant $C(\hat{\mu}_9)$ found by a numerical integration is equal to 2.99. The densities $\pi(\mu)$ and $\hat{\pi}(\mu \mid \hat{\mu}_9)$ as well as the estimate $\hat{L}_{1000}(\hat{\mu}_9 \mid \mu)$ are shown in Figure 3.

4. Estimating the failure probability with the fragility function involving epistemic uncertainty

4.1. Prior density

The estimation of μ can be extended for the case of the epistemic f.f. $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$. As in the case of the aleatory f.f. $F(\boldsymbol{y})$, one can introduce the epistemic r.v.

$$M' = E_{\boldsymbol{X}}\left(F(\boldsymbol{\nu}(\boldsymbol{X} \mid \boldsymbol{\Xi}) \mid \boldsymbol{\Theta})\right) = \int_{\text{all } \boldsymbol{x}} F(\boldsymbol{\nu}(\boldsymbol{x} \mid \boldsymbol{\Xi}) \mid \boldsymbol{\Theta}) \mathrm{d}F_{\boldsymbol{X}}(\boldsymbol{x}) \cdot \qquad (12)$$

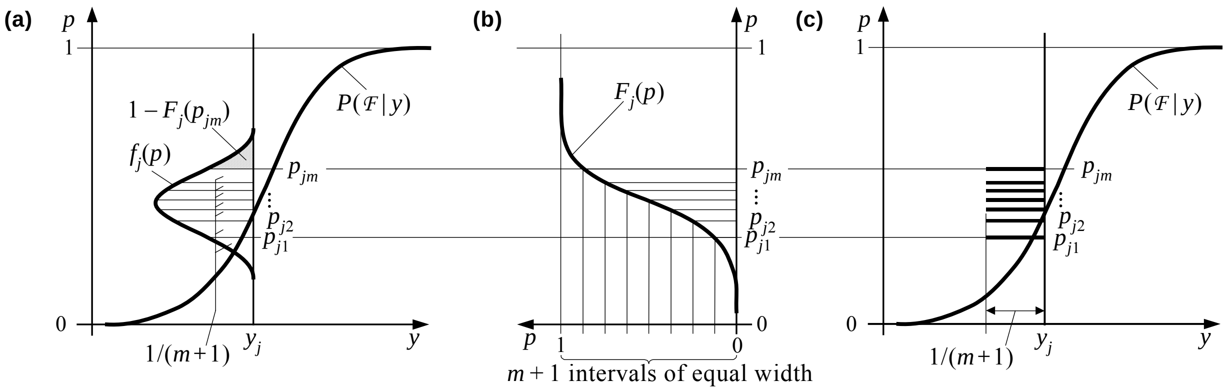

Figure 4. An approach to a discretisation of the continuous distribution of the epistemic random variable P_j

A value of M' is the failure probability corresponding to given values $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$ of $\boldsymbol{\Xi}$ and $\boldsymbol{\Theta}$. A density of M' can be applied as a natural prior $\pi(\mu)$ of μ [24].

4.2.Sample of new data

In case of the epistemic f.f. $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$, an incorporation of the new data **y** into updating $\pi(\mu)$ becomes non-trivial. The element $\boldsymbol{y}_j$ of **y** generates an epistemic r.v.

$$P_j = F(\boldsymbol{y}_j \mid \boldsymbol{\Theta}), \qquad (13)$$

which can be treated as imprecise observation with its own p.d.f. $f_j(p)$ and d.f. $F_j(p)$ (Figure 4a,b). Consequently, the epistemic f.f. $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$ requires to update $\pi(\mu)$ using a set of n imprecise "observations" $\{P_1, P_2, \dots, P_j, \dots, P_n\}$.

An updating of the prior $\pi(\mu)$ with the information expressed by the r.v.s P_j is a nontrivial problem. The posterior averaging approach mentioned in Introduction is not directly applicable to the present case. This approach was developed for a discrete distribution of a single uncertain datum [25, 28]. In principle, the posterior averaging could be applied by discretising the distributions of P_j in the traditional way. However, these distributions can be discretised and the prior $\pi(\mu)$ updated without using the posterior averaging. The heuristic principle of this discretisation is that it should yield m values p_{jk} of P_j and these values should have equal epistemic weights $w_k = 1/m$ (k = 1, 2, …, m). The equal weights w_k assure that none of p_{jk}s will be preferred to others. The equal w_ks is an analogy with the equal attitude towards elements of a sample collected by following a standard probability sampling scheme (e.g. [38, p. 106]).

The suggested principle of the discretisation is illustrated in Figure 4b,c. The values p_{jk} can be calculated by

$$p_{jk} = F_j^{-1}(k/(m+1)) \quad (k = 1, 2, \dots, m), \qquad (14)$$

where $F_j^{-1}(\cdot)$ is the inverse d.f. of P_j. The non-uniformly arranged values p_{jk} can be interpreted as ones of a r.v. with the probability masses w_k equal to $1/(m+1)$ (Figure 4c). The discretisation leads to a loss of the upper tail area $1 - F_j(p_{jm})$ (Figure 4a), and so w_ks do not strictly satisfy the condition $\sum_k w_k = 1$. However, this discrepancy will decrease when the number m increases.

After the transformation (14) is applied to all n elements of the sample **y**, a new sample consisting of $n \times m$ elements is obtained:

$$\boldsymbol{p} = \{(p_{jk}, k = 1, 2, \dots, m), j = 1, 2, \dots, n\}. \qquad (15)$$

When the same number m is applied to discretise each P_j, all elements of $\boldsymbol{p}$ will have equal epistemic weights approximately equal to $1/m$. Then the sample $\boldsymbol{p}$ defined by Eq (15) can be applied in place of the sample (7) to updating the prior $\pi(\mu)$.

4.3. Numerical implementation and recipes

The discretisation of the continuous probability distribution of P_j distorts to a degree the information expressed by this distribution. In addition, the discretisation raises the question about the number m of the discrete values p_{jk} related to a specific $\boldsymbol{y}_j$. One can expect that the larger is m the closer is the distribution of the probabilities p_{jk} (k = 1, 2, … , m) to the distribution of P_j (e. g., the closer is the mean $\bar{p}_{jk}$ of the values p_{jk} to the mean of P_j). However, the excessively large m will lead to an excessively large size $n \times m$ of the sample $\boldsymbol{p}$. This in turn can influence the results of the Bayesian updating of the prior density $\pi(\mu)$.

At present, one can say that the number m can be chosen adaptively. Criteria for the choice of m can be based on an interpretation of the quantities p_{jk} (k = 1, 2, …, m) as a statistical sample which can be denoted by

$$\boldsymbol{p}_j = \{p_{jk}, k = 1, 2, \dots, m\}. \qquad (16)$$

These criteria can be derived by comparing the empirical distribution of $\boldsymbol{p}_j$ to the distribution of P_j. As an illustration, let us assume three distributions of P_j having the same mean and different degrees of skewness (Table 2). Each of them was discretised and four samples $\boldsymbol{p}_j$ were created using the values of m = 10, 20, 50, and 100. Descriptive measures of $\boldsymbol{p}_j$ are given in Cols. 2 to 4 of Table 3. It follows from this table that the difference between the mean values $\bar{\boldsymbol{p}}_j$ of $\boldsymbol{p}_j$ and the distribution mean EP_j = 0.05 increases with the increase of the distribution skewness αP_j. The standard deviation $s\boldsymbol{p}_j$ and skewness $a\boldsymbol{p}_j$ of $\boldsymbol{p}_j$ is smaller than the corresponding values of the probability distributions σP_j and αP_j (Tables 2 and 3). This, probably, is due to the loss of the upper tail area of the distribution of P_j (Figure 4a).

Table 2. Three examples of the continuous probability distribution of the epistemic random variable P_j

Distribution type	Mean EP_j	St. dev. σP_j	Parameter μ	Parameter σ	Skewness αP_j
Normal	0.05	0.03	0.05	0.03	0
Lognormal	0.05	0.03	–3.149475	0.554513	2.02*
Lognormal	0.05	0.05	–3.342306	0.8325546	4.0*

**Calculated by the formula $\alpha P_j = (\exp\{\sigma^2\}+2)(\exp\{\sigma^2\}-1)^{-1/2}$

Table 3. Descriptive measures of the initial sample $\boldsymbol{p}_j$ and the adjusted sample $\boldsymbol{p}'_j$ resulting from the discretisation of three probability of the epistemic random variable P_j at different values of m

	Characteristics of the initial sample $\boldsymbol{p}_j$			Characteristics of the initial sample $\boldsymbol{p}'_j$			
m	Mean $\bar{\boldsymbol{p}}_j$	St. dv. $s\boldsymbol{p}_j$	Skew. $a\boldsymbol{p}_j$	p'_{j1}	Mean $\bar{\boldsymbol{p}}'_j$	St. dv. $s\boldsymbol{p}'_j$	Skew. $a\boldsymbol{p}'_j$
1	2	3	4	5	6	7	8
Case 1: P_j ~N(0.05, 0.03²), $\boldsymbol{p}'_j$ obtained by means of the transformation (17)							
10	0.05	0.02493	—(1)	0.001804	0.05	0.03	—(1)
20	0.05	0.02674	—	-0.006160	0.05	0.03	—
50	0.05	0.02827	—	-0.015642	0.05	0.03	—
100	0.05	0.02896	—	-0.022404	0.05	0.03	—
Case 2: P_j ~L(–3.149475, 0.554513), $\boldsymbol{p}'_j$ obtained by means of the transformation (17)							
10	0.04711	0.02162	0.814	0.013005	0.05	0.03	0.814
20	0.04807	0.02395	0.986	0.011080	0.05	0.03	0.986
50	0.04894	0.02625	1.211	0.009687	0.05	0.03	1.211
100	0.04935	0.02747	1.369	0.008960	0.05	0.03	1.369
Case 3: P_j ~L(–3.342306, 0.8325546), $\boldsymbol{p}'_j$ obtained by means of the transformation (17)							
10	0.04356	0.02984	1.185	-0.003506	0.05	0.05	1.185
20	0.04557	0.03435	1.451	-0.003497	0.05	0.05	1.451
50	0.04746	0.03930	1.824	-0.002294	0.05	0.05	1.824
100	0.04839	0.04220	2.109	-0.001304	0.05	0.05	2.109
Case 4: P_j ~L(–3.342306, 0.8325546), $\boldsymbol{p}'_j$ obtained by means of the transformation (19)(2 … 5)							
10	0.04356	0.02984	1.185	0.011633	0.05019(2)	0.03929	1.431
20	0.04557	0.03435	1.451	0.008815	0.05019(3)	0.04164	1.675
50	0.04746	0.03930	1.824	0.006352	0.05000(4)	0.04388	2.005
100	0.04839	0.04220	2.109	0.005081	0.05015(5)	0.04567	2.275

(1) Negligibly small value not exceeding 1×10^{-15}

(2) Δ_j = 0.29, (3) Δ_j = 0.22, (4) Δ_j = 0.14, (5) Δ_j = 0.11

Table 4. Results of the application of the goodness-of-fit tests to the sample $\boldsymbol{p}'_j$ obtained by discretising the lognormal distribution L(– 3.149475, 0.554513)

m	K-S d_{max}	p-value of d_{max}	Chi-square	p-value of chi-square
10	0.0826	1.0	too few values	too few values
20	0.0498	1.0	″	″
50	0.0259	1.0	0.0867	0.958
100	0.0158	1.0	0.1649	0.921

The deviation of the descriptive measures of $\boldsymbol{p}_j$ from the corresponding theoretical values can be eliminated or decreased by transforming $\boldsymbol{p}_j$. For instance, a simple linear transformation of the sample $\boldsymbol{p}_j$ will not change the type of distribution of $\boldsymbol{p}_j$, namely,

$$p'_{jk} = (\sigma P_j / s\boldsymbol{p}_j)(p_{jk} - \overline{\boldsymbol{p}}_j) + EP_j\,. \tag{17}$$

Eq (17) yields an adjusted sample

$$\boldsymbol{p}'_j = \{p'_{jk}, k = 1,2, \dots, k\}\,, \tag{18}$$

with the mean value and standard deviation precisely equal to the corresponding characteristics of P_j (Cols. 6 and 7, Table 3). At the same time, the transformation (17) leaves the skewness of $\boldsymbol{p}_j$ unchanged (Cols. 4 and 7, Table 3).

The transformation (17) is applicable to both symmetrical and skewed distributions. However, it can produce negative values of p'_{jk}, especially, in case of small probabilities (e.g., Cases 1 and 3, Table 3). As probability is limited by the interval [0, 1], Eq (17) is applicable only to the case where $p'_{j1} > 0$ (Case 2, Table 3).

The sample $\boldsymbol{p}_j$ can be adjusted to the distribution of P_j by applying the transformation

$$p'_{jk} = p_{jk}(1 + \Delta_j (p_{jk} - p_{j1})/p_{j1})\,, \tag{19}$$

where Δ_j is the adjustment factor, the value of which can be chosen adaptively. The transformation (19), strictly speaking, is non-linear; however, the departure from linearity is not large at small values of Δ_j. The transformation (19) makes the mean value $\overline{\boldsymbol{p}}'_j$ of the adjusted sample $\boldsymbol{p}'_j$ virtually equal to the distribution mean EP_j (Case 4, Col. 6, Table 3). At the same time, it makes standard $s\boldsymbol{p}'_j$ and skewness $a\boldsymbol{p}'_j$ of $\boldsymbol{p}'_j$ closer to the respective values σP_j and αP_j, especially in case of small values of m (Case 4, Cols. 7 and 8 in Table 3).

The minimum value of m can be chosen by applying goodness-of-fit tests to the samples $\boldsymbol{p}'_j$. For instance, Table 4 shows results of applying two standard tests to the sample $\boldsymbol{p}'_j$ obtained by discretising one of the lognormal distributions. One can conclude that $\boldsymbol{p}'_j$ fits the lognormal distribution quite well even at $m = 10$.

Further implementation problem is that the type of the probability distribution of P_j will in most cases be unknown. However, the probability p_{jk} following from Eq (14) is the quantile of the r.v. P_j with the level of $k/(m+1)$. In such a case the value p_{jk} can be estimated by the empirical quantile $\hat{p}_{j,k/(m+1)}$ computed for the sample

$$\boldsymbol{p}''_j = \{p_{j1}, p_{j2}, \dots, p_{js}, \dots, p_{jn_s}\}, \tag{20}$$

where the sample element p_{js} is obtained by sampling the value $\boldsymbol{\theta}_s$ of the parameter vector $\boldsymbol{\Theta}$ from $F_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and evaluating the f.f. $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$ for the pair $\boldsymbol{y}_j$ and $\boldsymbol{\theta}_s$:

$$p_{js} = F(\boldsymbol{y}_j \mid \boldsymbol{\theta}_s)\,. \tag{21}$$

With the sample $\boldsymbol{p}''_j$, the empirical quantile $\hat{p}_{j,k/(m+1)}$ is obtained in the standard way, namely, by ordering elements of $\boldsymbol{p}''_j$ and choosing the element with the number $[n_s \times k/(m+1)]+1$.

Two sets of the samples $\boldsymbol{p}'_j$ and $\boldsymbol{p}''_j$ can be combined into two samples

$$\boldsymbol{p}' = \{\boldsymbol{p}'_j, j = 1, 2, \dots, n\}\,, \tag{22}$$

$$\boldsymbol{p}'' = \{\boldsymbol{p}''_j, j = 1, 2, \dots, n\}\,. \tag{23}$$

The fist sample $\boldsymbol{p}'$ can be applied to updating the prior p.d.f. $\pi(\mu)$ instead of the initial sample $\boldsymbol{p}$ defined by Eq (15). The simulated sample $\boldsymbol{p}''$ can be used to control the quality of information represented by the sample $\boldsymbol{p}'$ obtained by means of discretisation. It is natural to expect that descriptive measures of $\boldsymbol{p}'$ and $\boldsymbol{p}''$ will be relatively close to each other.

4.4. Second example: the use of epistemic fragility function

4.4.1. Prior density of failure probability

The first example described in Sec 3.4 will now be expanded by introducing an epistemic f.f. $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$. This is expressed by a d.f. of a normal distribution, $F(y \mid \Theta_1, \Theta_2)$, with uncertain mean Θ_1 and uncertain variance Θ_2. They are assumed to be independent and distributed as indicated in Table 5. The gamma prior G(18, 14.962) of the precision Θ_2^{-1} is equivalent to an inverted gamma prior IG(18, 14.962) of the variance Θ_2 [39, p.20]. The unique mode of IG(18, 14.962) is 0.7875 (kPa)^2 (e.g. [40, p.119]). This value is equal to the “crisp” value of the corresponding f.f. parameter θ_2 (Sec 3.4.1).

As in the previous example (Sec 3.4.2), the prior density $\pi(\mu)$ was fitted using a nested loop simulation procedure. This generated the sample $\{\mu_1, \mu_2, \dots, \mu_l, \dots, \mu_{n_l}\}$, the lth element of which, μ_l, is an estimate of the mean value $E_{\boldsymbol{X}}(p_i(\boldsymbol{v}(\boldsymbol{X} \mid \xi_l) \mid \theta_l))$ at the given values ξ_l and $\boldsymbol{\theta}_l = (\theta_{1l}, \theta_{2l})^{\mathrm{T}}$ (see Eq (7)). The latter value was sampled from the distributions given in Table 5. The sample size n_l was assumed to be equal to 1000.

Table 5. Prior distributions of the f.f. parameters Θ_1 and Θ_2*

Parameter of f.f.	Type of prior	Parameters of prior distribution
Θ_1	Normal	7 kPa (mean), 0.77 kPa (sd. dev.)
Θ_2^{-1}	Gamma	18 (shape), 14.962 (kPa)$^{-2}$ (scale), 1.1362 (kPa)$^{-2}$ (mode)

* According to recommendations of Congdon [39, p. 19]

It was problematic to fit a widely known univariate probability distribution to the sample $\{\mu_1, \mu_2, \dots, \mu_l, \dots, \mu_{1000}\}$. Therefore this was transformed into the sample $\{-\ln\mu_1, -\ln\mu_2, \dots, -\ln\mu_{1000}\}$ and a gamma distribution Ga(0.1557, 17.4929) was fitted to the latter sample (Figure 5). The transformation $\psi = -\ln\mu$ was chosen intuitively. It implies that the prior $\pi(\mu)$ can be obtained from the p.d.f. $f_\Psi(\psi)$ of the r.v. Ψ~Ga($\alpha = 0.1557$, $\beta = 17.4929$) using the following density transformation [41, p. 26]:

$$\pi(\mu \mid -\alpha, \beta) = -\frac{1}{\mu} f_\Psi(\ln\mu \mid -\alpha, \beta)\,, \tag{24}$$

where α and β are the scale and shape parameters of the gamma distribution, respectively. The prior density $\pi(\mu)$ obtained using the transformation (24) is shown in Figure 6. It has a somewhat higher coefficient of variation that the prior density specified with the aleatory f.f. $F(y \mid \boldsymbol{\theta})$ (Sec 3.4.2).

Table 6. Descriptive measures of the samples $\boldsymbol{p}$, $\boldsymbol{p}'$, and $\boldsymbol{p}''$ used in the first and second examples

Sample size	Mean	Std.dev.	Skewness	Kurtosis	Minimum	Maximum	10$^{\text{th}}$ prc.	90$^{\text{th}}$ prc.
The sample $\boldsymbol{p}$ obtained by applying the crisp fragility function (Table 1)								
9	0.02048	$5.692\cdot10^{-8}$	1.78	2.02	$5.692\cdot10^{-8}$	$6.402\cdot10^{-3}$	—*	—
The sample $\boldsymbol{p}'$ obtained using the discretisation with m = 50 (Eq (22))								
450	0.013234	0.038145	5.2357	34.331	$5.60\cdot10^{-14}$	0.3724	$2.27\cdot10^{-7}$	0.03406
The sample $\boldsymbol{p}'$ obtained using the discretisation with m = 100 (Eq (22))								
900	0.013261	0.039008	5.5544	39.680	$3.89\cdot10^{-15}$	0.4356	$1.94\cdot10^{-7}$	0.03414
The sample $\boldsymbol{p}''$ obtained using the simulation (Eq (23))								
900 000	0.013197	0.040145	6.3105	55.820	0.0	0.9590	—	—

* Not calculated

Table 7. Descriptive measures of the simulated samples $\boldsymbol{p}''_j$ obtained with n_s = 100 000 and computed for the elements y_j of the initial sample $\mathbf{y}$

j	y_j (kPa)	Mean	St.dev.	Skew.	Kurt.	Minimum	Maximum	10$^{\text{th}}$ prc.	90$^{\text{th}}$ prc
1	3.767	$4.112\cdot10^{-3}$	$1.394\cdot10^{-2}$	9.16	136	$2.220\cdot10^{-16}$	$4.167\cdot10^{-1}$	$1.058\cdot10^{-6}$	$9.661\cdot10^{-3}$
2	4.276	$1.289\cdot10^{-2}$	$3.215\cdot10^{-2}$	5.58	45.56	$8.882\cdot10\text{-}^{16}$	$5.976\cdot10^{-1}$	$1.609\cdot10^{-5}$	$3.528\cdot10^{-2}$
3	4.160	$9.742\cdot10^{-3}$	$2.569\cdot10^{-2}$	6.00	53.49	$5.440\cdot10^{-15}$	$6.284\cdot10^{-1}$	$8.701\cdot10^{-6}$	$2.591\cdot10^{-2}$
4	3.944	$6.149\cdot10^{-3}$	$1.858\cdot10^{-2}$	7.61	94.53	$2.887\cdot10^{-14}$	$6.185\cdot10^{-1}$	$2.784\cdot10^{-6}$	$1.519\cdot10^{-2}$
5	4.916	$4.305\cdot10^{-2}$	$7.558\cdot10^{-2}$	3.39	15.37	$1.664\cdot10^{-11}$	$9.590\cdot10^{-1}$	$3.108\cdot10^{-4}$	$1.240\cdot10^{-1}$
6	2.920	$4.660\cdot10^{-4}$	$2.650\cdot10^{-3}$	19.82	767	0.0	$2.038\cdot10^{-1}$	$4.286\cdot10^{-9}$	$6.625\cdot10^{-4}$
7	4.791	$3.472\cdot10^{-2}$	$6.487\cdot10^{-2}$	3.64	17.73	$2.542\cdot10^{-12}$	$8.604\cdot10^{-1}$	$1.793\cdot10^{-4}$	$9.938\cdot10^{-2}$
8	4.032	$7.570\cdot10^{-3}$	$2.191\cdot10^{-2}$	6.90	70.77	$1.104\cdot10^{-13}$	$4.581\cdot10^{-1}$	$4.384\cdot10^{-6}$	$1.925\cdot10^{-2}$
9	2.294	$8.094\cdot10^{-5}$	$7.008\cdot10^{-4}$	29.7	1371	0.0	$5.130\cdot10^{-2}$	$4.418\cdot10^{-11}$	$6.435\cdot10^{-5}$

4.4.2. New information used for updating

The new information was represented by the sample $\boldsymbol{p}'$ obtained by clustering the nine samples $\boldsymbol{p}'_j$ (j = 1, 2, … , 9; see Eqs (18) and (22)). The sample $\boldsymbol{p}'_j$ was computed by transforming the corresponding sample $\boldsymbol{p}_j$ by means of Eq (19). The linear transformation (17) was not applied because it produced negative elements p'_{jk} of $\boldsymbol{p}'_j$ in all nine cases. The sample $\boldsymbol{p}_j$ is a result of discretising the r.v. P_j with the d.f. $F_j(p)$ into a set of m quantiles p_{jk} defined by Eq (14). As the d.f. $F_j(p)$ is not known in the present case, the values p_{jk} were estimated by the empirical quantiles $\hat{p}_{j,k/(m+1)}$ computed for the samples $\boldsymbol{p}''_j$, each consisting of 100 000 simulated values p_{js} of the r.v. $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$ (i.e., n_s = 100 000, see Eq (20)). The discretisation of P_j was carried out using two sets of the quantiles $\hat{p}_{j,k/(m+1)}$, namely, m = 50 and m = 100.

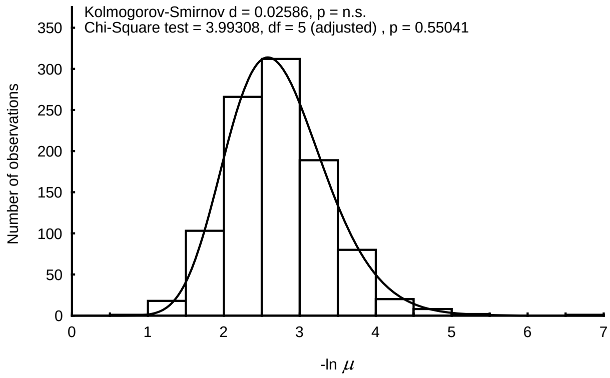

Figure 5. Histogram of the sample $\{-\ln\mu_1, -\ln\mu_2, \dots, -\ln\mu_{1000}\}$ and density of the gamma distribution Ga(0.1557, 17.4929) fitted to this sample

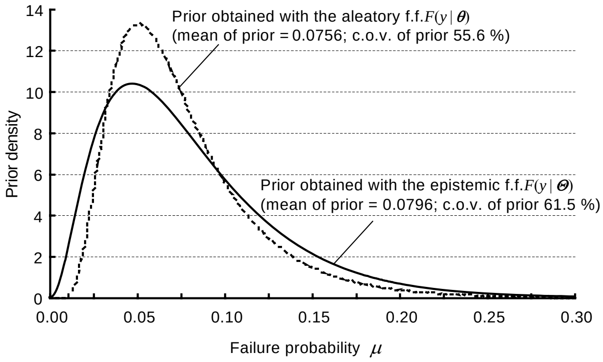

Figure 6. Lognormal prior density $\pi(\mu\,|\,-2.71691, 0.519298)$ (dashed line) and transformed gamma prior density $\pi(\mu\,|\,-0.1557, 17.4929)$ (solid line)

The simulated samples $\boldsymbol{p}''_j$ were combined into the sample $\boldsymbol{p}''$ consisting of 900 000 elements (Eq (23)). Descriptive measures of $\boldsymbol{p}''$ and $\boldsymbol{p}''_j$ are given in Tables 6 and 7, respectively.

The samples $\boldsymbol{p}''_j$ can be used to control the results of the discretisation expressed by the samples of quantiles, $\boldsymbol{p}_j$ and $\boldsymbol{p}'_j$. For instance, descriptive measures of the latter samples computed for the case $m = 100$ are given in Tables 8 and 9. Descriptive measures of $\boldsymbol{p}_j$ differ from the ones of $\boldsymbol{p}''_j$ to a relatively large extend (compare Tables 7 and 8). The transformation (19) produced the samples $\boldsymbol{p}'_j$ which are closer to $\boldsymbol{p}''_j$ in terms of their mean values, standard deviations, and skewnesses (compare Tables 7 and 9). Larger differences in the descriptive measures were obtained only in the cases of $j = 6$ and $j = 9$, namely, in cases of a relatively large skweness of the samples $\boldsymbol{p}''_6$ and $\boldsymbol{p}''_9$ (lines 6 and 9, Table 7). One can conclude that in case of highly skewed samples $\boldsymbol{p}''_j$ the transformation (19) should be replaced by a more sophisticated one which will yield better adjustment of the samples $\boldsymbol{p}_j$ to the simulated samples $\boldsymbol{p}''_j$.

Table 8. Descriptive measures of the samples $\boldsymbol{p}_j$ obtained using the transformation (14) with $m = 100$

j	y_j (kPa)	Mean	St.dev.	Skew.	Kurt.	Minimum	Maximum	10th prc.	90th prc
1	3.767	$3.415\cdot10^{-3}$	$9.041\cdot10^{-3}$	4.51	23.90	$2.210\cdot10^{-9}$	$6.396\cdot10^{-2}$	$1.026\cdot10^{-6}$	$8.622\cdot10^{-3}$
2	4.276	$1.153\cdot10^{-2}$	$2.505\cdot10^{-2}$	3.73	16.43	$7.397\cdot10^{-8}$	$1.624\cdot10^{-1}$	$1.568\cdot10^{-5}$	$3.203\cdot10^{-2}$
3	4.160	$8.606\cdot10^{-3}$	$1.948\cdot10^{-2}$	3.89	17.79	$3.195\cdot10^{-8}$	$1.285\cdot10^{-1}$	$8.488\cdot10^{-6}$	$2.366\cdot10^{-2}$
4	3.944	$5.273\cdot10^{-3}$	$1.309\cdot10^{-2}$	4.28	21.57	$7.107\cdot10^{-9}$	$9.057\cdot10^{-2}$	$2.711\cdot10^{-6}$	$1.365\cdot10^{-2}$
5	4.916	$4.069\cdot10^{-2}$	$6.671\cdot10^{-2}$	2.73	8.51	$3.934\cdot10^{-6}$	$3.755\cdot10^{-1}$	$3.054\cdot10^{-4}$	$1.157\cdot10^{-1}$
6	2.920	$3.219\cdot10^{-4}$	$1.165\cdot10^{-3}$	5.95	40.74	$1.638\cdot10^{-12}$	$9.409\cdot10^{-3}$	$4.141\cdot10^{-9}$	$5.742\cdot10^{-4}$
7	4.791	$3.258\cdot10^{-2}$	$5.651\cdot10^{-2}$	2.92	9.87	$1.935\cdot10^{-6}$	$3.274\cdot10^{-1}$	$1.752\cdot10^{-4}$	$9.242\cdot10^{-2}$
8	4.032	$6.544\cdot10^{-3}$	$1.572\cdot10^{-2}$	4.11	19.85	$1.347\cdot10^{-8}$	$1.064\cdot10^{-1}$	$4.247\cdot10^{-6}$	$1.750\cdot10^{-2}$
9	2.294	$4.403\cdot10^{-5}$	$1.929\cdot10^{-4}$	6.86	52.74	$3.886\cdot10^{-15}$	$1.659\cdot10^{-3}$	$4.219\cdot10^{-11}$	$5.402\cdot10^{-5}$

Table 9. Descriptive measures of the samples $\boldsymbol{p}'_j$ obtained by transforming the samples $\boldsymbol{p}_j$ by means of Eq (19) (the latter samples result from the discretisation of continuous distributions of r.v.s P_j at $m = 100$)

j	y_j (kPa)	Δ_j	Mean	St.dev.	Skew.	Kurt.	Minimum	Maximum	10th prc.	90th prc
1	3.767	0.51	$4.153\cdot10^{-3}$	$1.250\cdot10^{-2}$	5.33	33.4	$2.210\cdot10^{-9}$	$9.658\cdot10^{-2}$	$1.026\cdot10^{-6}$	$9.215\cdot10^{-3}$
2	4.276	0.28	$1.283\cdot10^{-2}$	$3.008\cdot10^{-2}$	4.19	21.0	$7.397\cdot10^{-8}$	$2.079\cdot10^{-1}$	$1.568\cdot10^{-5}$	$3.380\cdot10^{-2}$
3	4.160	0.35	$9.830\cdot10^{-3}$	$2.442\cdot10^{-2}$	4.45	23.6	$3.195\cdot10^{-8}$	$1.735\cdot10^{-1}$	$8.488\cdot10^{-6}$	$2.519\cdot10^{-2}$
4	3.944	0.43	$6.211\cdot10^{-3}$	$1.725\cdot10^{-2}$	4.99	29.5	$7.107\cdot10^{-9}$	$1.295\cdot10^{-1}$	$2.712\cdot10^{-6}$	$1.453\cdot10^{-2}$
5	4.916	0.16	$4.327\cdot10^{-2}$	$7.416\cdot10^{-2}$	2.95	10.2	$3.934\cdot10^{-6}$	$4.356\cdot10^{-1}$	$3.054\cdot10^{-4}$	$1.214\cdot10^{-1}$
6	2.920	0.95	$4.680\cdot10^{-4}$	$2.068\cdot10^{-3}$	7.26	58.9	$1.638\cdot10^{-12}$	$1.835\cdot10^{-2}$	$4.141\cdot10^{-9}$	$6.075\cdot10^{-4}$
7	4.791	0.18	$3.490\cdot10^{-2}$	$6.363\cdot10^{-2}$	3.18	12.0	$1.935\cdot10^{-6}$	$3.864\cdot10^{-1}$	$1.752\cdot10^{-4}$	$9.712\cdot10^{-2}$
8	4.032	0.39	$7.598\cdot10^{-3}$	$2.021\cdot10^{-2}$	4.75	26.7	$1.347\cdot10^{-8}$	$1.479\cdot10^{-1}$	$4.247\cdot10^{-6}$	$1.863\cdot10^{-2}$
9	2.294	1.60	$8.142\cdot10^{-5}$	$4.595\cdot10^{-4}$	8.35	75.0	$3.886\cdot10^{-15}$	$4.313\cdot10^{-3}$	$4.219\cdot10^{-11}$	$5.683\cdot10^{-5}$

For the case $m = 100$, clustering the nine samples $\boldsymbol{p}'_j$ resulted in a sample $\boldsymbol{p}'$ containing 900 elements and having descriptive measures presented in Table 6. This table contains also descriptive measures of the sample $\boldsymbol{p}'$ obtained with $m = 50$ and consisting of 450 elements. Results presented in Table 6 indicate that the samples $\boldsymbol{p}'$ are relatively close to the sample $\boldsymbol{p}''$ as regards their mean values, standard deviations and measures of skewness. Consequently, the samples $\boldsymbol{p}'$ can be used for updating the prior p.d.f. $\pi(\mu)$.

4.4.3. Results of updating by means of Bayesian bootstrap

The samples $\boldsymbol{p}'$ containing 450 and 900 elements were used to calculate the respective likelihood function estimates $L_B(\hat{\mu}_{450} \mid \mu)$ and $L_B(\hat{\mu}_{900} \mid \mu)$ by means of Eq (9). Then Eq (10) was used to obtain the approximations of posterior density, $\hat{\pi}(\mu \mid \hat{\mu}_{450})$ and $\hat{\pi}(\mu \mid \hat{\mu}_{900})$. The normalizing constants $C(\hat{\mu}_{450})$ and $C(\hat{\mu}_{900})$ found by a numerical integration are equal to 3.08868 and 3.089694, respectively. As in the previous example, the number of bootstrap replications, B, necessary to generate the sample $\{\hat{\mu}'_{n1}, \hat{\mu}'_{n2}, \ldots, \hat{\mu}'_{nB}\}$ was taken to be equal to 1000 and the bandwidth w was chosen to be 0.1. Figure 7 shows the graphs of the functions $\pi(\mu)$, $L_B(\hat{\mu}_{450} \mid \mu)$, and $\hat{\pi}(\mu \mid \hat{\mu}_{450})$.

The difference between the likelihood function estimates $L_B(\hat{\mu}_{450} \mid \mu)$ and $L_B(\hat{\mu}_{900} \mid \mu)$ is slight (Figure 8). This results in a slight difference between the posterior densities, $\hat{\pi}(\mu \mid \hat{\mu}_{450})$ and $\hat{\pi}(\mu \mid \hat{\mu}_{900})$ (Figure 9). The random fluctuation of differences shown in Figures 8 and 9 is due to the application of the stochastic simulation to the sampling of bootstrap samples. The small difference between $L_B(\hat{\mu}_{450} \mid \mu)$ and $L_B(\hat{\mu}_{900} \mid \mu)$ can be explained by looking at the terms in the sum of Eq (9). The means values of the samples $\boldsymbol{p}'$ consisting of 450 and 900 elements are approximately equal, namely, $\hat{\mu}_{450}$ = 0.013234 and $\hat{\mu}_{900}$ = 0.013261 (Table 6). The mean values of the bootstrap samples $\hat{\mu}'_{450,b}$ and $\hat{\mu}'_{900,b}$ seem to be relatively close, no matter what is the size of $\boldsymbol{p}'$. An indirect confirmation of this are the virtually equal mean values of the samples consisting of $\hat{\mu}'_{450,b}$ and $\hat{\mu}'_{900,b}$:

$B^{-1}\sum_{b=1}^{B}\hat{\mu}'_{450,b} = 0.0132398$ (st.dev. of $\hat{\mu}'_{450,b} =$ 0.00182),

$B^{-1}\sum_{b=1}^{B}\hat{\mu}'_{900,b} = 0.0132397$ (st.dev. of $\hat{\mu}'_{900,b} =$ 0.00128).

The results just mentioned allow us to conclude that doubling the discretisation number m from 50 to 100 and so the size $n \times m$ of the sample $\boldsymbol{p}'$ does not tangibly influence the posterior density $\hat{\pi}(\mu \mid \hat{\mu}_{n \times m})$. Thus the number m should be chosen mainly for reasons of the best approximation of the continuous epistemic probability distribution by the sample $\boldsymbol{p}'$.

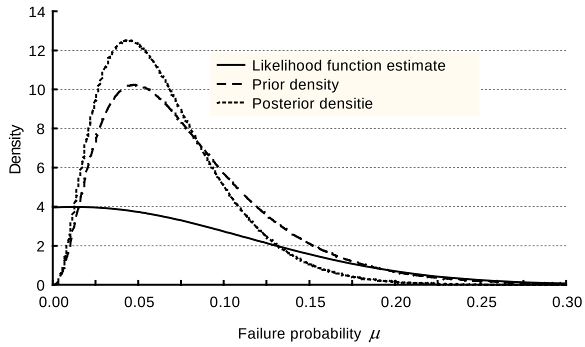

Figure 7. Likelihood function estimate $L(\hat{\mu}_{450} \mid \mu)$ (solid line), prior density $\pi(\mu)$ (dash and line) and estimate of posterior density $\hat{\pi}(\mu \mid \hat{\mu}_{450})$ (dotted line) obtained with the bandwidth $w = 0.1$

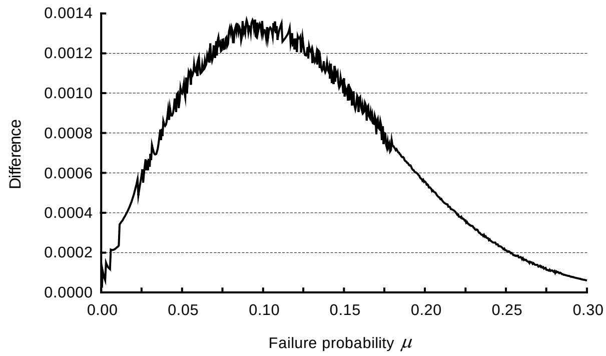

Figure 8. Values of the difference $|L(\hat{\mu}_{450} \mid \mu) - L(\hat{\mu}_{900} \mid \mu)|$

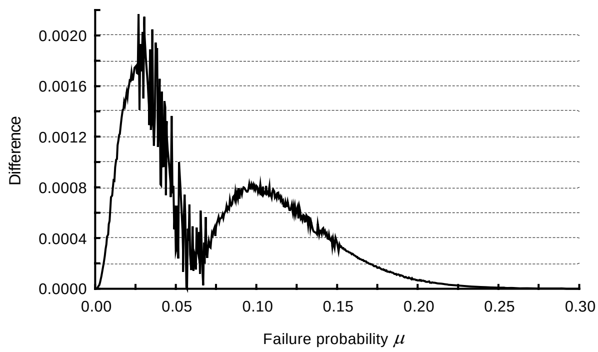

Figure 9. Values of the difference $|\hat{\pi}(\mu \mid \hat{\mu}_{450}) - \hat{\pi}(\mu \mid \hat{\mu}_{900})|$

The approximation of the posterior density, $\hat{\pi}(\mu \mid \hat{\mu}_{450})$, expresses the updated epistemic uncertainty in the failure probability $P(\mathcal{F} \mid \mathcal{AS})$. Figure 7 indicates that $\hat{\pi}(\mu \mid \hat{\mu}_{450})$ is more accurate that the prior density $\pi(\mu)$. The degree of "accuracy" can be expressed by the ranges of non-conservative and conservative percentiles given in Table 10. The new nine experimental records of the blast wave represented by the sample $\mathbf{y}$ decreased the uncertainty expressed by the prior density $\pi(\mu)$. One can anticipate that the conservative percentiles derived from $\hat{\pi}(\mu \mid \hat{\mu}_{450})$ will be better understandable for the decision maker that the densities themselves. Thus the decision concerning the potential failure event $\mathcal{F}$ can be made by applying these percentiles.

Table 10. Pairs of approximate percentiles derived from the prior densities $\pi(\mu)$ and the approximation of the posterior densities, $\hat{\pi}(\mu \mid \hat{\mu}_9)$ and $\hat{\pi}(\mu \mid \hat{\mu}_{450})$, obtained using the crisp fragility function $F(\boldsymbol{y} \mid \boldsymbol{\theta})$ and uncertain fragility function $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$

Density characteristic	Densities obtained with $F(\boldsymbol{y} \mid \boldsymbol{\theta})$		Densities obtained with $F(\boldsymbol{y} \mid \boldsymbol{\Theta})$	
	Prior $\pi(\mu)$	Posterior estimate $\hat{\pi}(\mu \mid \hat{\mu}_9)$	Prior $\pi(\mu)$	Posterior estimate $\hat{\pi}(\mu \mid \hat{\mu}_{450})$
5th percentile	0.02813	0.0263	0.0205	0.0185
95th percentile	0.1553	0.1218	0.174	0.134
Range	0.1272	0.0955	0.1535	0.1155
1st percentile	0.0197	0.0186	0.0115	0.0105
99th percentile	0.2012	0.1593	0.236	0.175
Range	0.2015	0.1407	0.2245	0.1645

5. Conclusions

Estimating an imprecise failure probability by applying scarce and uncertain information related to a potential failure in an abnormal situation has been considered. Two sources of information were applied to this estimating: (i) a small-size statistical sample consisting of experimental observations of characteristics of abnormal situation and (ii) fragility function used to express aleatory and epistemic uncertainty related to the potential failure. Estimating the failure probability was formulated as a problem of Bayesian inference. Epistemic uncertainty in the failure probability was expressed by means of Bayesian prior and posterior distributions. The central problem of estimating was Bayesian updating with imprecise data. Such data were an intermediate result of probability estimating. The imprecise data were represented by a set of continuous epistemic probability distributions of the fragility function values related to elements of the small-size sample.

The Bayesian updating with the set of continuous epistemic distributions is possible by discretising these distributions. The discretisation yields a new sample which can be used for updating. This sample consists of fragility function values, each of which has equal epistemic weight. Such a discretisation can be obtained by dividing the range of the inverse distribution function of each epistemic distribution into equal intervals. In case where the continuous epistemic distributions are highly skewed, an additional transformation of the discrete distribution can improve the discretisation.

The proposed approach is also applicable to the case where the continuous epistemic distributions are not available in the explicit form and must be represented by simulated samples of fragility functions values. In this case, the discretisation can be obtained using percentiles of the simulated samples. Such a simulation will be possible for the fragility function, the values of which can be evaluated with a relatively small computational effort.

Estimating the failure probability using the sample resulting from the discretisation was illustrated by two examples. The probability of failure due to an accidental explosion was considered in these examples. The probability was estimated using a fragility function which expresses the aleatory uncertainty only and a fragility function which quantifies both aleatory and epistemic uncertainty.

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