INVESTIGATION OF THE GERBER-SHIU DISCOUNTED PENALTY FUNCTION ON FINITE TIME HORIZON

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Abstract. In this paper, the classical risk model with exponential claim sizes is considered. The explicit expression of the Gerber-Shiu discounted penalty function $\psi(x, \delta, t)$ and discounted moments $\phi_m(x, \delta, t)$ on finite time horizon is obtained, where $\delta > 0$ is the force of interest, x - the initial reserve. Also the expression of the Gerber-Shiu discounted penalty function on time interval $[t_1, t_2]$ is derived. The dependance of discounted penalty function on the main parameters x, θ, λ, δ, μ is presented in diagrams, where $\lambda > 0$ is the parameter of Poisson process, $\theta > 0$ is the safety loading coefficient and μ is the parameter of claim distribution.

Keywords: classical risk model, time to ruin, Gerber-Shiu discounted penalty function.

1. Introduction and main result

In 1957, Sparre Andersen [1] proposed a classical risk model, which was applied to the risk business of an insurance company. In this model, the insurer's surplus process $\{U(t),\, t \geq 0\}$ is given by

$$U(t) = x + ct - \sum_{i=1}^{N(t)} Y_i,$$

where x is insurer's initial surplus, c is the rate of premium income per unit time. $\{N(t),\, t \geq 0\}$ is a Poisson process with parameter λ. Individual claims $Y_1, Y_2, \ldots$ form a sequence of independent and identically distributed positive random variables, with Y_i representing the amount of the *i*-th individual claim.

Define T_x to be the time to ruin, so that

$$T_x = \inf\{t > 0 : U(t) < 0 | U(0) = x\}.$$

If $U(t) \geq 0$ for all t, then $T_x = \infty$. The function

$$\psi(x) = P(T_x < \infty)$$

is called the ruin probability, and the function

$$\psi(x, t) = P(T_x < t)$$

is called the finite time ruin probability.

In 1998, Gerber and Shiu [2] proposed, instead of the probability of ruin in the classical risk model, to analyze the discounted penalty function

$$\psi(x, \delta) = E(e^{-\delta T_x} \mathbf{I}_{\{T_x < \infty\}}), \quad \delta \geq 0.$$

This function describes the expectation of the present value of future ruin and is called the Gerber-Shiu discounted penalty function. Here, δ is the force of interest and T_x is the ruin time. In this case the penalty at the moment T_x is accepted to be unit.

In 2000, a number of fundamental results about the properties of $\psi(x, \delta)$ were presented by Willmot and Lin [3]. For example, one of results says that the Gerber-Shiu discounted penalty function $\psi(x, \delta)$ satisfies the defective renewal equation. We formulate this assertion below.

Theorem 1. [3] *Assume that claim sizes $Y_1, Y_2, \ldots$ in the classical model have absolutely continuous distribution with a d.f. H(y) and a finite mean EY. Let N(t) be a Poisson process with parameter λ, and let the premium rate $c = \lambda EY(1 + \theta)$, with $\theta > 0$. Then $\psi(x, \delta)$ satisfies the defective renewal equation:*

$$\psi(x, \delta) = \widehat{\phi} \int_0^x \psi(x - y, \delta) dF(y) + \widehat{\phi} \bar{F}(x),$$

where

$$\bar{F}(x) = \frac{\int_0^\infty e^{-\rho y}\bar{H}(x+y)dy}{\int_0^\infty e^{-\rho y}\bar{H}(y)dy},$$

$$\widehat{\phi} = \frac{\int_0^\infty e^{-\rho y}\bar{H}(y)dy}{EY(1+\theta)}$$

and ρ is the unique non-negative root of Lundberg's equation

$$\lambda\int_0^\infty e^{-\rho y}H(y)dy = \lambda+\delta-c\rho.$$

In 2005, using the double Laplace transform, Garcia [5] obtained the expression for the density function of the time to ruin in the case where individual claims have mixed exponential distribution and Erlang(2) distribution.

In 2003, Drekic and Willmot [4] obtained the probability density function of the time of ruin T_x in the classical risk model with exponential claim sizes with parameter μ. By inversion of the associated Laplace transform they have got an expression of this function:

$$f(t) = \frac{e^{-\mu x}e^{-\lambda(2+\theta)t}}{t\sqrt{1+\theta}}\sum_{n=0}^{\infty}\frac{(n+1)(\mu x)^n}{n!(\sqrt{1+\theta})^n} \times I_{n+1}(2\lambda t\sqrt{1+\theta}), \quad (1)$$

where

$$I_p(y) = \sum_{k=0}^{\infty}\frac{(y/2)^{2k+p}}{k!(k+p)!} \quad (2)$$

is the modified Bessel function of the first kind of order p.

In this paper, we analyze the classical risk model with exponential claim sizes. Our purpose is to find the explicit expression of the Gerber-Shiu discounted penalty function on the finite time horizon:

$$\psi(x,\delta,t) = E(e^{-\delta T_x}\mathbf{I}_{\{T_x<t\}}), \quad \delta\geq 0, \quad t>0.$$

Also we derive the expressions for discounted moments on the finite time horizon

$$\phi_m(x,\delta,t) = E(T_x^m e^{-\delta T_x}\mathbf{I}_{\{T_x<t\}}),$$

$$m=1,2,..., \quad t>0, \quad \delta\geq 0$$

and for Gerber-Shiu discounted penalty function on time interval $[t_1,t_2]$:

$$\gamma(\delta,x,t_1t_2) = E(e^{-\delta T_x}\mathbf{I}_{\{t_1<T_x<t_2\}})$$

where

$$\delta\geq 0, \quad t_1,t_2>0.$$

Finally, all derived expressions are examined for various parameter choices (see Fig. 1 - 3). The following statement is the main result of this paper.

Theorem 2. *Assume the claim sizes $Y_1, Y_2, ...$ in the classical risk model have an exponential distribution with parameter μ. Let, in addition, λ be the parameter of a Poisson process, $c=\lambda EY(1+\theta)$ be a premium income rate with positive parameter θ, let $x\geq 0$ be an initial surplus and let non-negative δ be the force of interest. Define $\nu=\delta+\lambda(2+\theta)$ and $\kappa=\lambda\sqrt{1+\theta}$. Then the following statements hold.*
(a) *The Gerber-Shiu discounted penalty function on the finite time horizon has the following form:*

$$\psi(x,\delta,t) = \phi e^{-(1-\phi)\mu x} - e^{-\nu t}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!} \times\left(\phi e^{-(1-\phi)\mu x} - \frac{\lambda}{\nu}e^{-\mu x} \times\sum_{n=0}^{j-1}\frac{n+1}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times\sum_{k=0}^{[\frac{j-n+1}{2}]-1}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\right),$$

where

$$\phi = \frac{\lambda(\nu-\sqrt{\nu^2-4\kappa^2})}{2\kappa^2}.$$

(b) *The Gerber-Shiu discounted penalty function on time interval $[t_1,t_2]$ is given by*

$$\gamma(x,\delta,t_1,t_2) = \frac{\lambda}{\nu}e^{-\mu x}\sum_{n=0}^{\infty}\frac{(n+1)}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times\sum_{k=0}^{\infty}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k} \times\sum_{j=0}^{n+2k}\left(t_1^j e^{-\nu t_1} - t_2^j e^{-\nu t_2}\right).$$

(c) *Discounted moments on the finite time horizon are*

$$\begin{aligned}\phi_m(x,\delta,t) &= \frac{\nu^{m+1}}{\lambda}e^{\mu x}\phi_m(x,\delta) \\ &- \frac{\lambda}{\nu^{m+1}}e^{-(\mu x+\nu t)}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!} \\ &\times \left(\phi_m(x,\delta) - \sum_{n=0}^{j-1}\frac{n+1}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n\right. \\ &\times \left.\sum_{k=0}^{[\frac{j-n+1}{2}]-1}\frac{(2k+n+m)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\right),\end{aligned}$$

where

$$\begin{aligned}\phi_m(x,\delta) &= \frac{e^{-\frac{\mu\theta x}{1+\theta}}}{1+\theta}\frac{(m-1)!}{\lambda^m} \\ &\times \sum_{i=0}^{m-1}\left(\frac{\lambda x}{c}\right)^{m-1-i}\frac{(m-i+\frac{\lambda x}{c})}{(m-1-i)!} \\ &\times \sum_{l=0}^{i}\binom{m}{i-l}\binom{m+l-1}{l}\theta^{-m-l}.\end{aligned}$$

2. Proof of Theorem 2

In this section, applying (1) and (2) we will derive the expression of the Gerber-Shiu discounted penalty function on the finite time horizon as well as on time interval $[t_1, t_2]$. Also we will find the expression for discounted ruin moments. We start our proof of the theorem from part (c).

Step (1). According to (1) and (2),

$$\begin{aligned}\phi_m(x,\delta,t) &= E(T_x^m e^{-\delta T_x}\mathbf{I}_{\{T_x<t\}} \qquad (3)\\ &= \int_0^t u^m e^{-\delta u}f(u)du \\ &= \frac{e^{-\mu x}}{\sqrt{1+\theta}}\sum_{n=0}^{\infty}\frac{(n+1)(\mu x)^n}{n!(\sqrt{1+\theta})^n} \\ &\times \int_0^t u^{m-1}e^{-u(\delta+\lambda(2+\theta))} \\ &\times I_{n+1}(2\lambda u\sqrt{1+\theta})du,\end{aligned}$$

where

$$\begin{aligned}I_{n+1}(2\lambda u\sqrt{1+\theta}) &= \sum_{k=0}^{\infty}u^{2k+n+1} \qquad (4)\\ &\times \frac{(\lambda\sqrt{1+\theta})^{2k+n+1}}{k!(k+n+1)!}.\end{aligned}$$

Substitution of (4) into (3) yields

$$\begin{aligned}\phi_m(x,\delta,t) &= \frac{e^{-\mu x}}{\sqrt{1+\theta}} \\ &\times \sum_{n=0}^{\infty}\frac{(n+1)(\mu x)^n}{n!(\sqrt{1+\theta})^n} \\ &\times \sum_{k=0}^{\infty}\frac{(\lambda\sqrt{1+\theta})^{2k+n+1}}{k!(k+n+1)!} \\ &\times \int_0^t e^{-u(\delta+\lambda(2+\theta))}u^{2k+n+m}du.\end{aligned}$$

Since $\nu = \delta+\lambda(2+\theta)$, $\kappa = \lambda\sqrt{1+\theta}$ and

$$\begin{aligned}\int_0^t e^{-\nu u}u^{2k+n+m}du &= \frac{(2k+n+m)!}{\nu^{2k+n+m+1}} \\ &\times \left(1-e^{-\nu t}\sum_{j=0}^{2k+n+m}\frac{(\nu t)^j}{j!}\right),\end{aligned}$$

we get the explicit expression of $\phi_m(x,\delta,t)$:

$$\begin{aligned}\phi_m(x,\delta,t) &= \frac{\lambda}{\nu^{m+1}}e^{-\mu x} \qquad (5)\\ &\times \sum_{n=0}^{\infty}\frac{(n+1)}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \\ &\times \sum_{k=0}^{\infty}\frac{(2k+n+m)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k} \\ &\times \left(1-e^{-\nu t}\sum_{j=0}^{2k+n+m}\frac{(\nu t)^j}{j!}\right).\end{aligned}$$

Next, we change the order of the summation in (5) and our expression becomes

$$\begin{aligned}\phi_m(x,\delta,t) &= \frac{\lambda}{\nu^{m+1}}e^{-\mu x}\alpha(x) \qquad (6)\\ &- \frac{\lambda}{\nu^{m+1}}e^{-(\mu x+\nu t)}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!} \\ &\times \left(\alpha(x) - \sum_{n=0}^{j-1}\frac{n+1}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n\right. \\ &\times \sum_{k=0}^{[\frac{j-n+1}{2}]-1}\frac{(2k+n+m)!}{k!(k+n+1)!} \\ &\times \left.\left(\frac{\kappa}{\nu}\right)^{2k}\right),\end{aligned}$$

where

$$\alpha(x) = \sum_{n=0}^{\infty} \frac{(n+1)}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times \sum_{k=0}^{\infty} \frac{(2k+n+m)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}.$$

Note that

$$\phi_m(x,\delta) = \lim_{t\to\infty}\phi_m(x,\delta,t) = \frac{\lambda}{\nu^{m+1}}e^{-\mu x} \times \sum_{n=0}^{\infty} \frac{(n+1)}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times \sum_{k=0}^{\infty} \frac{(2k+n+m)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}.$$

Since (e.g. Drekic and Willmot, [4], equation (3.11))

$$\phi_m(x,\delta) = \frac{e^{-\frac{\mu\theta x}{1+\theta}}}{1+\theta}\frac{(m-1)!}{\lambda^m} \times \sum_{i=0}^{m-1}\left(\frac{\lambda x}{c}\right)^{m-1-i}\frac{(m-i+\frac{\lambda x}{c})}{(m-1-i)!} \times \sum_{l=0}^{i}\binom{m}{i-l}\binom{m+l-1}{l}\theta^{-m-l},$$

we have that

$$\alpha(x) = \frac{\nu^{m+1}}{\lambda}e^{\mu x}\phi_m(x,\delta). \qquad (7)$$

Finally, substituting (7) in into (6) yields

$$\phi_m(x,\delta,t) = \frac{\nu^{m+1}}{\lambda}e^{\mu x}\phi_m(x,\delta) - \frac{\lambda}{\nu^{m+1}}e^{-(\mu x+\nu t)}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!} \times \left(\phi_m(x,\delta) - \sum_{n=0}^{j-1}\frac{n+1}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times \sum_{k=0}^{[\frac{j-n+1}{2}]-1}\frac{(2k+n+m)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\right).$$

Step (2). It is evident that

$$\phi_0(x,\delta,t) = E(e^{-\delta T_x}\mathbf{I}_{\{T_x<t\}}) = \psi(x,\delta,t).$$

Hence, if we take $m = 0$ in equation (5), we get the expression of the Gerber-Shiu discounted penalty function on the finite time horizon

$$\psi(x,\delta,t) = \frac{\lambda}{\nu}e^{-\mu x}\sum_{n=0}^{\infty}\frac{(n+1)}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times \sum_{k=0}^{\infty}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k} \times \left(1 - e^{-\nu t}\sum_{j=0}^{2k+n}\frac{(\nu t)^j}{j!}\right).$$

Analogously as in Step(1), changing the order of summation in the last expression we get

$$\psi(x,\delta,t) = \frac{\lambda}{\nu}e^{-\mu x}\beta(x) - \frac{\lambda}{\nu}e^{-(\mu x+\nu t)}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!}\left(\beta(x) - \sum_{n=0}^{j-1}\sum_{k=0}^{\left[\frac{j-n+1}{2}\right]-1}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\right), \qquad (8)$$

where

$$\beta(x) = \sum_{n=0}^{\infty}\frac{(n+1)}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times \sum_{k=0}^{\infty}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}.$$

Since

$$\psi(x,\delta) = \lim_{t\to\infty}\psi(x,\delta,t) = \frac{\lambda}{\nu}e^{-\mu x}\beta(x)$$

and (see e.g. Willmot and Lin, [3], equation (4.1.12))

$$\psi(x,\delta) = \phi e^{-\mu(1-\phi)x},$$

we obtain that

$$\beta(x) = \sum_{n=0}^{\infty}\frac{(n+1)}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n \times \sum_{k=0}^{\infty}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k} = \frac{\phi\nu}{\lambda}e^{\mu\phi x}, \qquad (9)$$

whith ϕ defined in the statement of Theorem 2.

Substituting (9) into (8) yields

$$\begin{aligned}\psi(x,\delta,t) &= \phi e^{-(1-\phi)\mu x} - e^{-\nu t}\sum_{j=0}^{\infty}\frac{(\nu t)^j}{j!}\\ &\times \Bigg(\phi e^{-(1-\phi)\mu x} - \frac{\lambda}{\nu}e^{-\mu x}\\ &\times \sum_{n=0}^{j-1}\frac{n+1}{n!}\left(\frac{\mu\lambda x}{\nu}\right)^n\\ &\times \sum_{k=0}^{\left[\frac{j-n+1}{2}\right]-1}\frac{(2k+n)!}{k!(k+n+1)!}\left(\frac{\kappa}{\nu}\right)^{2k}\Bigg).\end{aligned}$$

Step (3). In this step the expression of $\gamma(\delta,x,t_1,t_2)$ will be obtained. According to (1) and (2) we have

$$\begin{aligned}\gamma(\delta,x,t_1,t_2) &= E(e^{-\delta T_x}\mathbf{I}_{\{t_1<T_x<t_2\}}) \qquad (10)\\ &= \int_{t_1}^{t_2} e^{-\delta u}f(u)du\\ &= \frac{e^{-\mu x}}{\kappa}\sum_{n=0}^{\infty}\frac{(n+1)(\mu x)^n}{n!(\kappa)^n}\\ &\times \sum_{k=0}^{\infty}\frac{(\lambda\kappa)^{2k+n+1}}{k!(k+n+1)!}\\ &\times \int_{t_1}^{t_2} e^{-u\nu}u^{2k+n}du.\end{aligned}$$

Since

$$\begin{aligned}\int_{t_1}^{t_2} e^{-\nu u}u^{2k+n}du &= -\frac{(2k+n)!}{\nu^{2k+n+1}}\\ &\times \sum_{j=0}^{2k+n}\left(t_2^j e^{-\nu t_2} - t_1^j e^{-\nu t_1}\right),\end{aligned}$$

it follows from (10) that

$$\begin{aligned}\gamma(\delta,x,t_1,t_2) &= \lambda e^{-\mu x}\sum_{n=0}^{\infty}\frac{(n+1)(\mu x)^n}{n!(\kappa)^n}\\ &\times \sum_{k=0}^{\infty}\frac{(\lambda\kappa)^{2k+n}}{k!(k+n+1)!}\frac{(2k+n)!}{\nu^{2k+n+1}}\\ &\times \sum_{j=0}^{n+2k}\left(t_1^j e^{-\nu t_1} - t_2^j e^{-\nu t_2}\right).\end{aligned}$$

3. Graphs

In this section, we present several plots of the functions whose expressions were derived in this paper.

In graphs I, II and III dependance of the function $\psi(x,\delta,t)$ on main parameters is presented. As we may note, in all cases the function decreases in parameter x and increases in parameter t. Such a behavior can be explained by the dependence of the value of future ruin on the size of initial capital x. The bigger initial capital of insurance company the less probability to go bankrupt in future. Looking more closely at the main parameter settings in these examples, we may examine the known function in more detail. While the fixed value of the claim intensity λ remains small (graphs I, III), we observe that the value of future bankruptcy is less than in the cases where this parameter is large enough (graph II). Moreover, comparing these graphs, we see that the value of future bankruptcy is visibly smaller when λ is very small. If we take $\delta = 0$ (graph IV), we get the plot of ruin probability $\psi(x,t)$ with decreasing failure rate behavior by x and increasing by t.

Further, with parameter value $m = 1$, we get plots of the discounted mean $\phi_1(x,\delta,t)$ on finite time horizon (graphs V - VIII), whose tendency of behavior is similar to that of the function $\psi(x,\delta,t)$. Moreover, in the case $\delta = 0$ (graph VIII) we have plots of the mean of time to ruin, where the increase of the initial capital x causes the decrease of the function $\phi_1(x,\delta,t)$ with all t.

Finally, in the last two graphs we may observe plots of the discounted penalty function on time interval $[t_1,t_2]$. Comparing these two plots (graphs IX and IX), we note that the lager time interval, the value of future bankruptcy lager. Moreover, if we look more closely, large value of the initial capital x has the positive impact on function's $\gamma(x,\delta,t_1,t_2)$ behavior. As we can see, the values of future bankruptcy are less when initial capital remains bigger and increase with small values of initial capital x.

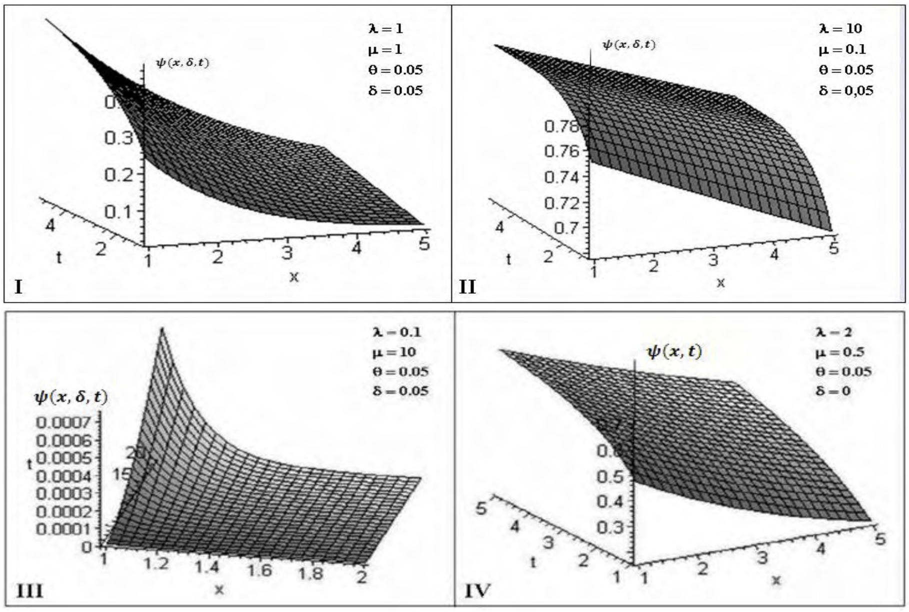

Fig.1. $\psi(x,\delta,t)$ dependance on parameters x, t, λ, μ, and δ.

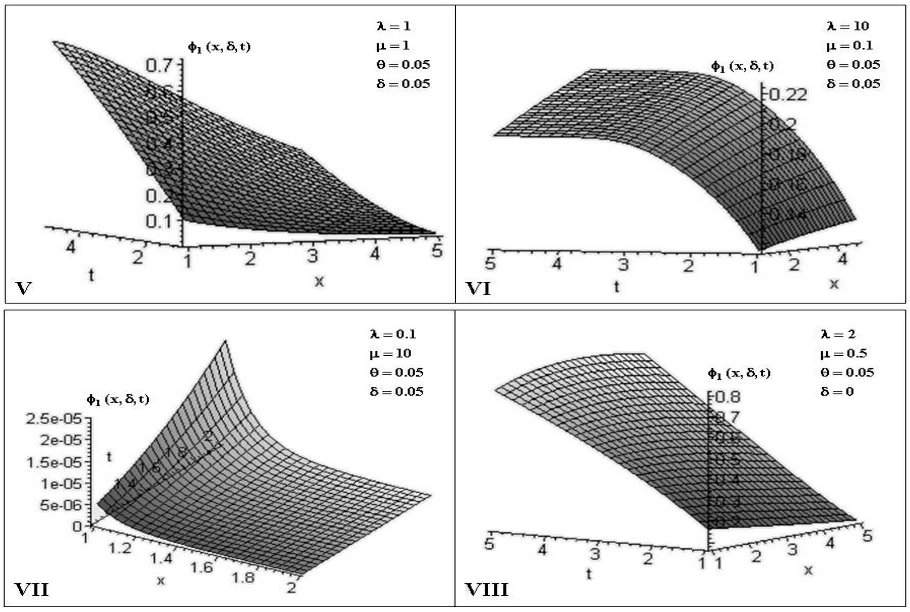

Fig.2. $\phi_1(x,\delta,t)$ dependance on parameters x, t, λ, μ, and δ.

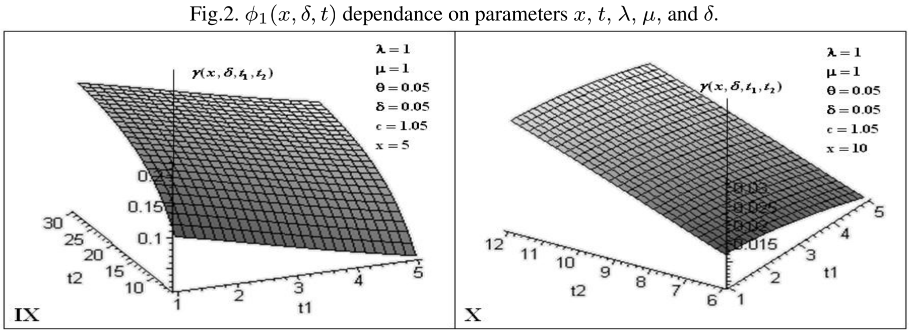

Fig.3. $\gamma(x,\delta,t_1,t_2)$ dependance on parameters x, t_1 and t_2.

4. Conclusions

The main goal of this article was to get the expressions of:

1. the Gerber-Shiu discounted penalty function on finite time interval;
2. discounted moment on finite time horizon;
3. the Gerber-Shiu discounted penalty function on time interval $[t_1, t_2]$.

To derive all these expressions, the known probability density function of ruin time has used. Also all mentioned functions were examined for various parameters settings and presented graphically.

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Received October 2009.