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INVESTIGATION OF A MULTIDIMENSIONAL AUTOMATIC CONTROL SYSTEM WITH DELAYS AND CHAIN FORM STRUCTURE

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Abstract. The dynamics of the forced synchronization system with delays, composed of n $(n \in N)$ oscillators joint into a chain, is studied. The investigation is based on the use of Jordan's form, eigenvalues and eigenvectors of the matrix, which describes the structure of the internal links of the system.

Keywords: synchronization system, differential equation, delayed argument.

1. Formulation of the problem

The automatic control systems are used in various processes of production and in the networks of transmitting and distributing of the information. Often the delays of the fed signals in such systems must the evaluated. Despite the great achievements in the area of projection and implementation of control systems, the works devoted to the exact analytical investigation of such systems are relevant [1-2]. In the present work, the exact theoretical investigation of the concrete linear multidimensional delay system is carried out.

Let us consider a multidimensional delay system described by the following matrix differential equation

$$Dx(t) = B_0 x(t) + B_1 x(t-\tau) + z(t)\,; \qquad (1)$$

here D is the generalized differential operator (applicable to generalized functions), $B_0 = \mathrm{diag}(0, -\kappa, \ -\kappa, \ \ldots, -\kappa)$, κ is the coefficient, $B_1 = \frac{\kappa}{2} B$,

$$B = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 0 & 1 & & 0 & \\ & 1 & 0 & 1 & & \\ & & & \ddots & & \\ & 0 & & 1 & 0 & 1 \\ & & & & 2 & 0 \end{pmatrix} \qquad (2)$$

is the n-th order matrix, $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ is the desired vector function (T denotes operation of transposition), $z(t)$ is the vector function, depending on the initial conditions, τ is the constant delay.

As an example of a control system described by the equation (1), the forced synchronization system of the communication network, composed of n oscillators joined into a chain, can be pointed out [1,2]. In this case, the symbol $x_i(t)$ $\left(i = \overline{1,n}\right)$ in (1) stands for the phase of the i-th oscillator.

We will investigate the dynamics of the system.

2. Step responses matrix of the system

2.1. The solution of the matrix equation

We solve the matrix differential equation with delayed argument (1) by applying method of "steps". The interval $0 < t < +\infty$ is divided into subintervals. The lengths of these subintervals are equal to the delay τ. The differential equation (1) in each subinterval is solved separately. The solution, obtained in some subinterval is used as initial function, solving the equation in the next subinterval. Applying the Laplace transform, we write down the solution of the equation (1) as follows [2]:

$$x(t) \div \sum_{l=0}^{L} \left(A^{-1} B_1 e^{-p\tau}\right)^l A^{-1} Z(p), \ \ 0 < t < (L+1)\tau\,;$$

here $A = pE - B_0$, E is the identity matrix of the n-th order,

$$A^{-1} = \frac{1}{p+\kappa}\,\mathrm{diag}\left(\frac{p+\kappa}{p}, 1, 1, \ldots, 1\right), \ Z(p) \div z(p),$$

$Z(p)$ is the Laplace transform of the function $z(t)$ (sign $\div$ links function with its Laplace transform), $L = 0, 1, 2, \ldots$.

Taking (2) into account, we obtain

$$x(t) \div \sum_{l=0}^{L} \left(\frac{\kappa}{2}\right)^{l} e^{-pl\tau} \left(A^{-1}\right)^{l} B^{l} A^{-1} Z(p),$$
$$0 < t < (L+1)\tau . \tag{3}$$

We will find the step responses matrix $h(t) = \left(h_{ij}(t)\right)$ of the system; here $h_{ij}(t)$ ($i, j = \overline{1,n}$) is the response of the i-th oscillator phase to a unit jump in the j-th oscillator oscillation phase. The set of all step responses $h_{ij}(t)$ ($i, j = \overline{1,n}$) form the step responses matrix $h(t)$ of the system.

Using the expression (3), we get

$$h(t) = (h_{ij}(t)) \div \sum_{l=0}^{L} \left(\frac{\kappa}{2}\right)^{l} e^{-pl\tau} \left(A^{-1}\right)^{l} B^{l} A^{-1},$$
$$0 < t < (L+1)\tau . \tag{4}$$

2.2. The *l*-th power of the matrix *B*

We will find the l-th power $(l \in N)$ of the matrix B in (4) by applying the expression $B^{l} = TJT^{-1}$ [4], where J is the Jordan's form of B, T is the transforming matrix. Matrices J and T can be found provided eigenvalues and eigenvectors of the matrix B are known. The eigenvalues of B are obtained by solving the characteristic equation

$$|B - \lambda E| = 0 . \tag{5}$$

Let us denote

$$D_n(\alpha) = \begin{vmatrix} \alpha & 0 & & & & \\ 1 & \alpha & 1 & & 0 & \\ & 1 & \alpha & 1 & & \\ & & & \ddots & & \\ & 0 & & 1 & \alpha & 1 \\ & & & & 2 & \alpha \end{vmatrix},$$

$$\Delta_n(\alpha) = \begin{vmatrix} \alpha & 1 & & & & \\ 1 & \alpha & 1 & & 0 & \\ & 1 & \alpha & 1 & & \\ & & & \ddots & & \\ & 0 & & 1 & \alpha & 1 \\ & & & & 1 & \alpha \end{vmatrix}; \tag{6}$$

here $\alpha \in R$. Then

$$|B - \lambda E| = D_n(-\lambda) .$$

From (6) it follows

$$D_n = \alpha\left(\Delta_{n-1} - \Delta_{n-3}\right), \tag{7}$$
$$\Delta_n = \alpha\Delta_{n-1} - \Delta_{n-2} \tag{8}$$
$$(\Delta_2 = \alpha^2 - 1,\ \Delta_1 = \alpha,\ \Delta_0 = 1);$$

here $D_n = D_n(\alpha)$, $\Delta_n = \Delta_n(\alpha)$.

Solving difference equation (8), we obtain

$$\Delta_n(\alpha) = U_n\left(\frac{\alpha}{2}\right); \tag{9}$$

here $U_n(x)$ is the n-th degree Chebyshev polynomial of the second kind [5]:

$$U_n(x) = \frac{\sin(n+1)\arccos x}{\sin \arccos x},\ \ 1 \le x \le 1.$$

Taking into account (9) and the relation $2T_n(x) = U_n(x) - U_{n-2}(x)$ (see [5]), from (7) we get

$$D_n(\alpha) = 2\alpha T_{n-1}\left(\frac{\alpha}{2}\right); \tag{10}$$

here $T_n(x)$ is the n-th degree Chebyshev polynomial of the first kind. All the roots of the polynomial $T_n(x)$ are distributed in the interval $[-1, 1]$ and can be found by using the relation:

$$x_{nk} = \cos\frac{(2k-1)\pi}{2n},\ \ k = 1, 2, ..., n . \tag{11}$$

This relation follows from the known equality [5]

$$T_n(x) = \cos n \arccos x,\ \ -1 < x < 1 . \tag{12}$$

Further two cases will be examined separately.

2.2.1. The order of the matrix *B* is an even number

Let n be an even number ($n = 2m,\ m \in \mathrm{N}$). Taking into account expressions (10) and (11), we find the roots of the characteristic equation (5) (the eigenvalues of the even order matrix B):

$$\lambda_{2k-1} = -2\cos\frac{(2k-1)\pi}{2n-2},\ \ k = 1,\ 2, 3, ...,\ n-1 . \tag{13}$$

The eigenvalues λ_{2k-1} $\left(k = 1,\ 2, ...,\ n-1; k \ne \frac{n}{2}\right)$ are simple eigenvalues $(l_{2k-1} = 1)$, while the eigenvalue λ_{n-1} $\left(k = \frac{n}{2}\right)$ is multiple $(l_{n-1} = 2)$; here l_i is a multiplicity of the eigenvalue λ_i. For simple eigenvalue λ_i there corresponds single Jordan's cell $J_1(\lambda_i)$ in the matrix *J*. For multiple eigenvalue λ_{n-1}, there corresponds single Jordan cell $J_2(\lambda_{n-1})$ in the matrix *J* as well, since the rank $r(B - \lambda_{n-1}E) = n-1$ and $n - r(B - \lambda_{n-1}E) = 1$. Taking this into account and applying the relation $\lambda_{2k-1} = -\lambda_{2n-2k-1}$ ($k = 1, 2, ..., \frac{n}{2} - 1$), we write down the Jordan's form of the matrix *B*:

$$J=\begin{pmatrix} -\lambda_{2n-3} & 0 & & & & & & & & \\ & -\lambda_{2n-5} & 0 & & & & & & & \\ & & \ddots & & & & & 0 & & \\ & & & -\lambda_{n+1} & 0 & & & & & \\ & & & & 0 & 1 & & & & \\ & & & & & 0 & 0 & & & \\ & & & & & & \lambda_{n+1} & 0 & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \lambda_{2n-5} & 0 \\ & & & & & & & & & \lambda_{2n-3} \end{pmatrix}. \tag{14}$$

Using the equality $J = T^{-1}BT$ we find the matrix T and the inverse matrix T^{-1}. We calculate the l-th power of the matrix B:

$$B^{l} = TJ^{l}T^{-1} = \frac{1}{2n-2}Q(l) = \frac{1}{2n-2}(q_{ij}(l))\ ; \tag{15}$$

here

$$q_{ij}(l) = \left(1+(-1)^{l+i+j}\right)p_{ij}\sum_{k=1}^{\frac{n-2}{2}}\lambda_{2n-2k-1}^{l-\alpha_j}\ \ \lambda_{n+2k-2}^{2}\ \cdot$$
$$\cdot\, U_{i-2}\left(\frac{\lambda_{2n-2k-1}}{2}\right)\ U_{j-2+\alpha_j}\left(\frac{\lambda_{2n-2k-1}}{2}\right), \tag{16}$$

$$\alpha_j = \begin{cases} 0, & \text{if } j \neq 1, \\ 1, & \text{if } j = 1, \end{cases}$$

$$p_{ij} = \begin{cases} 0, & \text{if } i = 1, \\ \dfrac{1}{2}, & \text{if } i \neq 1 \text{ and } j = n, \\ 1, & \text{if } i \neq 1 \text{ and } j \neq n, \end{cases}$$

$\lambda_{2k-1}\ (k = \overline{1, n-1})$ are eigenvalues of the matrix B (see (13)), $\lambda_{2k} = -2\cos\dfrac{(2k-2)\pi}{2n-2}$, $(k = \overline{1, n-2})$ are auxiliary numbers, satisfying the relation $\lambda_{2k} = -\lambda_{2n-2k-2}$, $k = \overline{1, \frac{n}{2}-1}$.

2.2.2. The order of the matrix *B* is an odd number

Now we shall examine the case, where n is an odd number $\left(n = 2m+1,\ m \in N\right)$. Taking (10) and (11) into account, we find the roots of the characteristic equation (5) (the eigenvalues of the odd order matrix B):

$$\lambda_k = \begin{cases} -2\cos\dfrac{(2\mathrm{k}-1)\pi}{2n-2}, & \text{if } k = 1,2,\ldots,\dfrac{n-1}{2}, \\ 0, & \text{if } k = \dfrac{n+1}{2}, \\ -2\cos\dfrac{(2\mathrm{k}-3)\pi}{2n-2}, & \text{if } k = \dfrac{n+3}{2}, \dfrac{n+5}{2}, \ldots, n. \end{cases} \tag{17}$$

Since all the eigenvalues λ_k $\left(k = \overline{1,n}\right)$ are simple, for eigenvalue λ_k, there corresponds single Jordan cell $J_1(\lambda_k)$ in the matrix J. Taking this into account, we write down the Jordan's form of the matrix B:

$$J = \mathrm{diag}\left(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\right). \tag{18}$$

Using the equality $J = T^{-1}BT$, we find the matrices T, T^{-1} and derive the expression of the l-th power $(l \in N)$ of the matrix B:

$$B^{l} = TJ^{l}T^{-1} = \frac{1}{2n-2}G(l) = \frac{1}{2n-2}\left(g_{ij}(l)\right); \tag{19}$$

here

$$g_{ij}(l) = \left(1+(-1)^{l+i+j}\right)p_{ij}\sum_{k=1}^{\frac{n-1}{2}}\lambda_k^{l-\alpha_j}\,\lambda_{\frac{n+1}{2}-k}^{2}\ \cdot$$
$$\cdot\, U_{i-2}\left(\frac{\lambda_k}{2}\right)U_{j-2+\alpha_j}\left(\frac{\lambda_k}{2}\right), \tag{20}$$

$$\alpha_j = \begin{cases} 0, & \text{if } j \neq 1, \\ 1, & \text{if } j = 1, \end{cases} \qquad p_{ij} = \begin{cases} 0, & \text{if } i = 1, \\ \dfrac{1}{2}, & \text{if } i \neq 1 \text{ and } \mathrm{j} = \mathrm{n}, \\ 1, & \text{if } i \neq 1 \text{ and } \mathrm{j} \neq \mathrm{n}, \end{cases}$$

λ_k $\left(k = \overline{1,n}\right)$ are the eigenvalues of the odd order matrix B (defined by (17)).

2.3. The step responses of the system

2.3.1. The order of the matrix *B* is an even number

Substituting (15) into (4) and implementing the inverse Laplace transform, we obtain the step responses matrix of the system: $h(t) = (h_{ij}(t))$; here

$$h_{ij}(t) = \begin{cases} e^{-\kappa t}1(t) + u_{ij}(t), & \text{if } i = j > 1, \\ u_{ij}(t), & \text{if } 1 < i \neq j > 1, \\ v_{ij}(t), & \text{if } i > 1 \text{ and } j = 1, \end{cases} \tag{21}$$

$$u_{ij}(t)=\frac{1}{2n-2}\sum_{l=1}^{L}\frac{1}{2^l}\,q_{ij}(l)\,\frac{\kappa^l(t-l\tau)^l}{l!}\cdot$$
$$\cdot e^{-\kappa(t-l\tau)}\,1(t-l\tau),\ i>1,\ j>1,\ 0<t<(L+1)\tau,$$

$$v_{ij}(t)=\frac{1}{2n-2}\sum_{l=1}^{L}\frac{1}{2^l}\,q_{ij}(l)\left[1-\sum_{\nu=0}^{l-1}\frac{\kappa^\nu(t-l\tau)^\nu}{\nu!}\cdot\right.$$
$$\left.\cdot e^{-\kappa(t-l\tau)}\right]1(t-l\tau),\ i>1, j=1,\ 0<t<(L+1)\tau,$$

$q_{ij}(l)$ is defined by (16); $1(t)=\begin{cases}1, \text{ if } t\geq 0,\\ 0, \text{ if } t<0,\end{cases}$ is the unit function.

Taking into account derived expressions, the step responses matrix for the forced synchronization system composed of n (n is an arbitrary even number) oscillators joined into a chain can be written down.

For example, if $n=4$, we get

$$J=\begin{pmatrix}-\lambda_5 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\lambda_5\end{pmatrix}=\begin{pmatrix}-a & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & a\end{pmatrix},$$

$$B^l=\frac{1}{6}(q_{ij})=\frac{1}{6}\begin{pmatrix}0 & 0 & 0 & 0\\ a_1 & a_2 & a_3 & a_2\\ a_2 & a_3 & a_4 & a_3\\ 2a_1 & 2a_2 & 2a_3 & 2a_2\end{pmatrix},$$

$a=\sqrt{3}$, $a_1(l)=a^{l-1}(1-(-1)^l)$, $a_2(l)=a^l(1+(-1)^l)$, $a_3(l)=a^{l+1}(1-(-1)^l)$, $a_4(l)=a^{l+2}(1+(-1)^l)$, $l\in N$, $l\geq 2$,

$$h(t)=(h_{ij}(t))=\begin{pmatrix}1(t) & 0 & 0 & 0\\ h_1^{(1)} & \alpha+h_2 & h_3 & h_2\\ h_2^{(1)} & h_3 & \alpha+h_4 & h_3\\ 2h_1^{(1)} & 2h_2 & 2h_3 & \alpha+2h_2\end{pmatrix},$$

$\alpha(t)=e^{-\kappa t}1(t)$,

$$h_i(t)=\frac{1}{6}\sum_{l=1}^{L}\frac{1}{2^l}\,a_i(l)\frac{\kappa^l(t-l\tau)^l}{l!}\,\mathrm{e}^{-\kappa(t-l\tau)}\,1(t-l\tau),$$
$i=2,3,4,\quad 0<t<(L+1)\tau,$

$$h_i^{(1)}(t)=\frac{1}{6}\sum_{l=1}^{L}\frac{1}{2^l}\,a_i(l)\left[1-\sum_{\nu=0}^{l-1}\frac{\kappa^\nu(t-l\tau)^l}{\nu!}\cdot\right.$$
$$\left.\cdot e^{-\kappa(t-l\tau)}\right]1(t-l\tau),\ i=1,2,\ 0<t<(L+1)\tau.$$

If $n=6$, we would obtain

$$J=\begin{pmatrix}-a & 0 & & & & \\ & -c & 0 & & 0 & \\ & & 0 & 1 & & \\ & & & 0 & 0 & \\ & 0 & & & c & 0\\ & & & & & a\end{pmatrix},$$

$$B^l=\frac{1}{10}(q_{ij})=$$
$$=\frac{1}{10}\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\\ a_1 & a_2 & a_6 & a_7 & a_8 & a_4\\ a_2 & a_6 & a_9 & a_{10} & a_{11} & a_8\\ a_3 & a_7 & a_{10} & a_{12} & a_{13} & a_{14}\\ a_4 & a_8 & a_{11} & a_{13} & a_{15} & a_{10}\\ 2a_5 & 2a_4 & 2a_8 & 2a_{14} & 2a_{10} & 2a_7\end{pmatrix},$$

$a_1(l)=(a^{l-1}d^2+c^{l-1}b^2)(1-(-1)^l)$,
$a_2(l)=(a^l d^2+c^l b^2)(1+(-1)^l)$,

$a_3(l)=(a^{l-1}+c^{l-1})(1-(-1)^l)$,
$a_4(l)=(a^l d-c^l b)(1+(-1)^l)$,

$a_5(l)=(a^{l-1}d-c^{l-1}b)(1-(-1)^l)$,
$a_6(l)=(a^{l+1}d^2+c^{l+1}b^2)(1-(-1)^l)$,

$a_7(l)=(a^l+c^l)(1+(-1)^l)$,
$a_8(l)=(a^{l+1}d-c^{l+1}b)(1-(-1)^l)$,

$a_9(l)=(a^{l+2}d^2+c^{l+2}b^2)(1+(-1)^l)$,
$a_{10}(l)=(a^{l+1}+c^{l+1})(1-(-1)^l)$,

$a_{11}(l)=(a^{l+2}d-c^{l+2}b)(1+(-1)^l)$,
$a_{12}(l)=(a^l b^2+c^l d^2)(1+(-1)^l)$,

$a_{13}(l)=(a^{l+1}b-c^{l+1}d)(1-(-1)^l)$,
$a_{14}(l)=(a^l b-c^l d)(1+(-1)^l)$,

$a_{15}(l)=(a^{l+2}+c^{l+2})(1+(-1)^l)$,

$a=2\cos\frac{\pi}{10}$, $c=2\cos\frac{3\pi}{10}$, $b=2\cos\frac{\pi}{5}$, $d=2\cos\frac{2\pi}{5}$, $l\in N$, $l\geq 2$,

$$h(t)=\left(h_{ij}(t)\right)=\begin{pmatrix} 1(t) & 0 & 0 & 0 & 0 & 0 \\ h_1^{(1)} & \alpha+h_2 & h_6 & h_7 & h_8 & h_4 \\ h_2^{(1)} & h_6 & \alpha+h_9 & h_{10} & h_{11} & h_8 \\ h_3^{(1)} & h_7 & h_{10} & \alpha+h_{12} & h_{13} & h_{14} \\ h_4^{(1)} & h_8 & h_{11} & h_{13} & \alpha+h_{15} & h_{10} \\ 2h_5^{(1)} & 2h_4 & 2h_8 & 2h_{14} & 2h_{10} & \alpha+2h_7 \end{pmatrix},$$

$$\alpha(t)=e^{-\kappa t}1(t),$$

$$h_i(t)=\frac{1}{10}\sum_{l=1}^{L}\frac{1}{2^l}a_i(l)\frac{\kappa^l(t-l\tau)^l}{l!}e^{-\kappa(t-l\tau)}1(t-l\tau),\quad i=2,4,\overline{6,15},\quad 0<t<(L+1)\tau,$$

$$h_i^{(1)}(t)=\frac{1}{10}\sum_{l=1}^{L}\frac{1}{2^l}a_i(l)\left[1-\sum_{\nu=0}^{l-1}\frac{\kappa^\nu(t-l\tau)^l}{\nu!}e^{-\kappa(t-l\tau)}\right]1(t-l\tau),\quad i=\overline{1,5},\quad 0<t<(L+1)\tau.$$

2.3.2. The order of the matrix *B* is an odd number

Substituting (19) into (4) and implementing necessary transformations, we find the step responses matrix of the system in the case where n is an odd number: $h(t)=(h_{ij}(t))$; here

$$h_{ij}(t)=\begin{cases} e^{-\kappa t}1(t)+\overline{u_{ij}}(t), \text{ if } i=j>1, \\ \overline{u_{ij}}(t), \text{ if } 1<i\neq j>1, \\ \overline{v_{ij}}(t), \text{ if } i>1 \text{ and } j=1, \end{cases} \tag{22}$$

$$\overline{u_{ij}}(t)=\frac{1}{2n-2}\sum_{l=1}^{L}\frac{1}{2^l}g_{ij}(l)\frac{\kappa^l(t-l\tau)^l}{l!}1(t-l\tau),$$
$$i>1,\ j>1,\ 0<t<(L+1)\tau,$$

$$\overline{v_{ij}}(t)=\frac{1}{2n-2}\sum_{l=1}^{L}\frac{1}{2^l}g_{ij}(l)\left[1-\sum_{\nu=0}^{l-1}\frac{\kappa^\nu(t-l\tau)^\nu}{\nu!}\cdot\right.$$
$$\left.\cdot e^{-\kappa(t-l\tau)}\right]1(t-l\tau),\ i>1, j=1, 0<t<(L+1)\tau,$$

$g_{ij}(l)$ is defined by (20).

Taking into account derived expressions, the step responses matrix for the forced synchronization system composed of n (n is an arbitrary odd number) oscillators joined into a chain can be written down.

For example, if $n=3$, we get $J=\mathrm{diag}(-\lambda_3,0,\lambda_3)=\mathrm{diag}(-a,0,a)$, $a=\sqrt{2}$,

$$B^l=\frac{1}{4}(q_{ij})=\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_1 \\ a_2 & a_3 & a_2 \end{pmatrix},$$

$a_1(l)=a^{l+1}(1-(-1)^1)$, $a_2(l)=a^{l+2}(1+(-1)^1)$,

$a_3(l)=a^{l+3}(1-(-1)^1)$, $l\in N$,

$$h(t)=(h_{ij}(t))=\begin{pmatrix} 1(t) & 0 & 0 \\ h_1^{(1)} & \alpha+h_2 & h_1 \\ h_2^{(1)} & h_3 & \alpha+h_2 \end{pmatrix},$$

$$\alpha(t)=e^{-\kappa t}1(t),$$

$$h_i(t)=\frac{1}{4}\sum_{l=1}^{L}\frac{1}{2^l}a_i(l)\frac{\kappa^l(t-l\tau)^l}{l!}\mathrm{e}^{-\kappa(t-l\tau)}1(t-l\tau)$$
$$i=1,2,3.$$

$$h_i^{(1)}(t)=\frac{1}{4}\sum_{l=1}^{L}\frac{1}{2^l}a_i(l)\left[1-\sum_{\nu=0}^{l-1}\frac{\kappa^\nu(t-l\tau)^\nu}{\nu!}\right.$$
$$\left.e^{-\kappa(t-l\tau)}\right]1(t-l\tau),\ i=1,2,\ 0<t<(L+1)\tau,$$

If $n=5$, we would obtain $J=\mathrm{diag}(-\lambda_5,-\lambda_4,0,\lambda_4,\lambda_5)=\mathrm{diag}(-a,-c,0,c,a)$,

$$a=2\cos\frac{\pi}{8},\quad c=2\cos\frac{3\pi}{8},$$

$$B^l=\frac{1}{8}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_6 & a_7 & a_4 \\ a_5 & a_4 & a_7 & a_8 & a_9 \\ 2a_{10} & 2a_5 & 2a_4 & 2a_9 & 2a_{11} \end{pmatrix},$$

$$a_1(l)=(a^{l-1}c^2+c^{l-1}a^2)(1-(-1)^l),$$
$$a_2(l)=(a^lc^2+c^la^2)(1+(-1)^l),$$

$$a_3(l)=(a^{l+1}c^2+c^{l+1}a^2)(1-(-1)^l),$$
$$a_4(l)=(a^{l+1}c^2-c^{l+1}a^2)(1+(-1)^l),$$

$$a_5(l)=(a^lc-c^la)(1-(-1)^l),$$
$$a_6(l)=(a^{l+2}c^2+c^{l+2}a^2)(1+(-1)^l),$$

$a_7(l) = (a^{l+2}c - c^{l+2}a)(1-(-1)^l)$,

$a_8(l) = (a^{l+2} + c^{l+2})(1+(-1)^l)$,

$a_9(l) = (a^{l+1} - c^{l+1})(1-(-1)^l)$,

$a_{10}(l) = (a^{l-1}c - c^{l-1}a)(1+(-1)^l)$,

$a_{11}(l) = (a^l + c^l)(1+(-1)^l)$, $l \in N$,

$\alpha(t) = e^{-\kappa t}1(t)$,

$$h_i(t) = \frac{1}{8}\sum_{l=1}^{L}\frac{1}{2^l}a_i(l)\frac{\kappa^l(t-l\tau)^l}{l!}e^{-\kappa(t-l\tau)}1(t-l\tau),$$

$i = \overline{2,9}, \quad 0 < t < (L+1)\tau,$

$$h_i^{(1)}(t) = \frac{1}{8}\sum_{l=1}^{L}\frac{1}{2^l}a_i(l)\left[1-\sum_{\nu=0}^{l-1}\frac{\kappa^l(t-l\tau)^\nu}{\nu!}e^{-\kappa(t-l\tau)}\right]\cdot$$

$\cdot 1(t-l\tau), \quad i = 1,2,5,10, \quad 0 < t < (L+1)\tau.$

4. Conclusions

1. The exact expressions for the step responses of the forced synchronization system with delays, composed of n $(n \in N)$ oscillators joined into a chain are obtained.
2. The obtained expressions can be applied to investigate the transient responses of the synchronization system, calculate statistical characteristics, examine behaviour of the system in the steady state.
3. The method of dynamic's investigation, used in the paper, can also be applied to other automatic control systems, described by the linear matrix differential equations of the first order with delayed argument.

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