ISSN 1392 – 124X INFORMATION TECHNOLOGY AND CONTROL, 2005, Vol. 34, No. 4

ON THE USE OF SPACE-FILLING CURVES IN CHANGING IMAGE DIMENSIONALITY

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Abstract. The paper describes a new generalized approach (idea) to solving the image dimensionality change problem. The proposed idea employs continuous space-filling curves (Hilbert, Peano, etc.), characterized by self-similar and self-avoiding geometrical construction. These curves, being applied to multi-dimensional images in the role of image scan trajectories, determine relatively high smoothness of generated one-dimensional image analogues.

To illustrate practical applicability and usefulness of the developed approach, some interesting task-oriented digital image processing techniques are discussed in brief, namely: hyperbolic image filtering in spaces of different dimensionality and efficient encoding of multi-dimensional "silhouette" images in the one-dimensional space.

Keywords: digital images, image smoothness and dimensionality, space-filling curves, image compression.

1. Indroduction

One of the main problems in the graphical data processing area – the efficient encoding (compression) of digital images. Successful attempts to solve the problem lead, in many cases, to the higher data compression ratios, to much better data transmission rates and to the more optimized use of data storage cells. Everybody, who is gone deep into the essence of the matter, is well aware that simultaneous improvement of the image compression effect (ratio) and the quality of a restored image (estimate) faces difficulties. Indeed, higher image compression ratios worsen the quality of image estimates, various undesirable effects (block structure, artifacts around sharp edges, etc.) reveal themselves and, vice versa, many attempts to improve the quality of restored images lessen the image compression effect.

Efficiency of a particular image processing (encoding) technique, for the most part, depends on the following three factors:

- the type of an image (binary images, grey-level images, colour images);
- dimensionality of an image (one-dimensional images, two-dimensional images, three-dimensional images);
- smoothness of an image.

For instance, hyperbolic image filtering techniques are under the deep influence of image dimensionality, i.e., the greater the number of dimensions of an image, the better compression ratios, as well as quality of reconstructed images, are obtained [1, 2]. Block-Truncation-Coding algorithms are very "sensitive" from the point of view of smoothness of an image, [3]. Productivity of fractal image coding procedures highly depends on the type and "fractal nature" of the image under processing, [4, 5, 6].

In the meanwhile, some algorithms are developed to process binary ("black-and-white") images, whereas others – to process grey-level (or, colour) images. Efficiency of the latter algorithms, as a rule, decreases considerably if one tries to apply them to images of another type. For instance, the JPEG Standard is highly efficient when used to compress two-dimensional grey-level images, but is absolutely inapplicable to "black-and-white" images, [7]. No version of JPEG is made known for three-dimensional images either.

Finally, there exist efficient specialized encoding procedures, oriented to process one-dimensional digital images (data sequences) and not fitted to use in a multi-dimensional case, and vice versa, [8]. So, rhetorically, why not to "move" the image under processing into another, more suitable, space preliminary?

The latter circumstance (changing image dimensionality) sometimes appears to be a right way out if someone makes his mind to increase efficiency and/or adaptability of a particular digital image processing technique. Unfortunately, investigations in the area are far from being advanced and need to be continued.

Bearing all this in mind, a new generalized approach (idea) to solving the image dimensionality change problem has been developed. The idea is based on the direct use of space-filling curves of various degrees (Hilbert, Peano, etc., [9]).

To demonstrate vitality, practical applicability and usefulness of the proposed generalized image dimensionality change approach, some new theoretical and experimental analysis results, associated with functioning of hyperbolic image filters in spaces of different dimensionality, as well as with coding technology of multi-dimensional binary „silhouette" images, are presented in the paper.

In parallels, the basic concepts and ideas, that are needed to describe, state and solve the image dimensionality change problem, are introduced and explored in the sections below.

2. The metric space of digital images, smoothness of an image

Let $S^d(n) = \{[X(m)] \,|\, m = (m_1, ..., m_d) \in I^d\}$ be the set of d-dimensional digital images (pixel arrays), where $I = \{0, 1, ..., N-1\}$, $N = r^n$, $n \in \mathrm{N}$ and r is an integer greater than 1; $X(m) \in \{0, 1, ..., 2^p - 1\}$, for all $m \in I^d$, $d \in \{1, 2, 3\}$ and p stands for the number of bits per pixel in $[X(m)]$ ($p = 1$, for binary images, and $p > 1$, for grey-level images).

The distance (metric) δ between any two elements of $S^d(n)$ - images $[X_1(m)]$ and $[X_2(m)]$ - is defined to be

$$\delta = \delta(X_1, X_2) = \left(\frac{1}{N^d} \sum_{m \in I^d} \left(X_2(m) - X_1(m)\right)^2\right)^{1/2}. \quad (1)$$

The (metric) formula is used every time when it is necessary to compare quality of a restored image against that of the original one.

Thus, $(S^d(n), \delta)$ is a finite metric space of d-dimensional digital images (at the n-th resolution level).

Let, now, $[Y_X(k)]$, $k = (k_1, ..., k_d) \in I^d$, be a discrete d-dimensional spectrum (DCT, Walsh, etc., [10]) of the image $[X(m)] \in S^d(n)$. It is well known that the spectral coefficients $Y_X(k)$ decrease in absolute value, as their serial numbers k (indices $k_1, ..., k_d$) increase, provided the basis vectors of the discrete transform in use are presented in a frequency order. Evidently, there exists a d-dimensional hyperbolic "surface"

$$z = z(x_1, ..., x_d) = C / (x_1 \cdot ... \cdot x_d)^{\alpha} \quad (C \geq 0,\ \alpha \geq 0), \quad (2)$$

which approximates the ordered array of spectral coefficients $\{|Y_X(k)| \,\big|\, k \in I^d,\ k_1^2 + ... + k_d^2 \neq 0\}$ in the mean squared error sense.

The quantity α (expression (2)), characterizing the shape of the hyperbolic "surface", i.e., the rate of decay of spectral coefficients (high frequency components of the image), as their serial numbers increase, is assumed to be the smoothness parameter (level, class) of the image $[X(m)] \in S^d(n)$. This assumption is intuitively understandable – the more intense manifestation of high frequency components in the discrete spectrum of the image, the more noticeable changes of pixel intensity values (sharp edges) are detected in the image, [11].

In what follows, we are to deal with digital images falling into the metric space $(S^d(n), \delta)$ ($n = \log_r N$, $r \in \{2, 3\}$), and no preliminary assertions will be made concerning their smoothness level.

3. The notion of a space-filling curve

Let us introduce the notion of a space-filling curve, using the terminology of iterated function systems (IFS), [12].

Suppose, (R^2, d) is a Euclidean metric space and $(H(\mathrm{R}^2), h)$ denotes the corresponding (fractal) space of nonempty closed subsets of the set R^2, with the Hausdorff metric $h = \max\{d(A, B), d(B, A)\}$, where $d(A, B) = \max_{x \in A}\{d(x, B)\}$, $d(x, B) = \min_{y \in B}\{d(x, y)\}$, for all $A, B \in H(\mathrm{R}^2)$; $d(B, A)$ is defined similarly.

Let $\omega_i : \mathrm{R}^2 \to \mathrm{R}^2$ (i=1, 2, ..., M) be a contractive affine transformation on the metric space (R^2, d), i.e., $\omega_i(x) = (a_i x_1 + b_i x_2 + e_i, c_i x_1 + d_i x_2 + f_i)$, for all $x = (x_1, x_2) \in \mathrm{R}^2$; $a_i, b_i, c_i, d_i, e_i, f_i \in \mathrm{R}$. Let, also,

$$s_i = \sup\{\hat{s}_i \,\big|\, d(\omega_i(x), \omega_i(y)) \leq \hat{s}_i \cdot d(x, y); x, y \in \mathrm{R}^2\}$$

indicate the contractivity factor for ω_i; $0 \leq s_i < 1$, i=1, 2, ..., M.

The metric space (R^2, d) together with a finite set of contractive affine transformations ω_i, i=1, 2, ..., M, is called an iterated function system, and is designated as IFS $\{\mathrm{R}^2; \omega_1, \omega_2, ..., \omega_M\}$.

Let us move the above affine transformations into the space $(H(\mathrm{R}^2), h)$. Then, $\omega_i : H(\mathrm{R}^2) \to H(\mathrm{R}^2)$, defined by $\omega_i(B) = \{\omega_i(x) | x \in B\}$, $\forall B \in H(\mathrm{R}^2)$, is a contractive affine transformation on $(H(\mathrm{R}^2), h)$, with contractivity factor s_i, i=1, 2, ..., M.

Combining $\omega_i : H(\mathrm{R}^2) \to H(\mathrm{R}^2)$, i=1, 2, ..., M, we produce a new transformation $W : H(\mathrm{R}^2) \to H(\mathrm{R}^2)$ on the fractal space $(H(\mathrm{R}^2), h)$, namely:

$$W(B) = \omega_1(B) \cup \omega_2(B) \cup ... \cup \omega_M(B) = \bigcup_{i=1}^{M} \omega_i(B),$$

for all $B \in H(\mathrm{R}^2)$. We note here that the latter transformation is also contractive, i.e., $h(W(B), W(C)) \leq s \cdot h(B,C)$, for all $B, C \in H(\mathrm{R}^2)$; $s = \max\{s_1, s_2, ..., s_M\}$.

The only fixed point $A \in H(\mathrm{R}^2)$ of W, such that

$$\begin{aligned} A = W(A) = \bigcup_{i=1}^{M} \omega_i(A) = \\ = \lim_{n\to\infty} (W \circ W \circ \ldots \circ W)(B) = \lim_{n\to\infty} (W^{\circ n}(B), \end{aligned} \tag{3}$$

$\forall B \in H(\mathrm{R}^2)$, represents an attractor (fractal, fractal image) of the IFS $\{\mathrm{R}^2; \omega_1, \omega_2, ..., \omega_M\}$, [12].

Iterated function systems, acting in a three-dimensional Euclidean space (R^3, d), are introduced in an analogous way.

Among the algorithms, applied to synthesizing attractors of IFS, we find the following ones – the deterministic algorithm, the random iteration algorithm and the escape time algorithm. The former two algorithms, roughly speaking, are based on the definition of the attractor of an IFS, i.e., explore the fact that the set A (expression (3)) is an attractive fixed point for $W : H(\mathrm{R}^2) \to H(\mathrm{R}^2)$, [12]. The third (escape time) algorithm rests on the application of shift dynamical systems, [13].

We shall comment the very first (deterministic) fractal synthesis algorithm in more detail, because it facilitates presentation and explanation of the notion of a space-filling curve, which, frankly speaking, is nothing but the attractor of a properly chosen IFS.

Consider the IFS $\{\mathrm{R}^2; \omega_1, \omega_2, ..., \omega_M\}$ and an arbitrary set $A_0 \in H(\mathrm{R}^2)$. Let us compute successively $A_n = W^{\circ n}(A_0)$, according to

$$\begin{aligned} A_n = \omega_1(A_{n-1}) \cup \omega_2(A_{n-1}) \cup \ldots \cup \omega_M(A_{n-1}) = \\ = \bigcup_{i=1}^{M} \omega_i(A_{n-1}), \end{aligned} \tag{4}$$

$n = 1, 2, ...$, (the deterministic algorithm is in action!). So, the sequence of closed nonempty sets $\{A_0, A_1, ..., A_n, ...\} \subset H(\mathrm{R}^2)$ is obtained. It is proved that the sequence converges to the attractor of the IFS in the Hausdorff metric, [12]. Consequently, for large values of n, the n-th term of the sequence A_n can be identified with the attractor of the IFS.

Now, consider a square region

$$\Re = \{x = (x_1, x_2) \in \mathrm{R}^2 \mid 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\}$$

– the "support" for two-dimensional images, described at arbitrary finite resolutions, namely: $r \times r$, $r^2 \times r^2, \ldots, r^n \times r^n$; $r \in \{2,3\}$.

To motivate the development, we take an IFS $\{\Re; \omega_1, \omega_2, ..., \omega_{r^2}\}$, where all the affine transformations are similitudes of scaling factor $1/r$, corresponding to the collage in Figure 1.

$\omega_1(\Re)$	$\omega_2(\Re)$
$\omega_3(\Re)$	$\omega_4(\Re)$

(a)

$\omega_1(\Re)$	$\omega_2(\Re)$	$\omega_3(\Re)$
$\omega_4(\Re)$	$\omega_5(\Re)$	$\omega_6(\Re)$
$\omega_7(\Re)$	$\omega_8(\Re)$	$\omega_9(\Re)$

(b)

Figure 1. The collage, obtained using a finite set of affine transformations (similitudes): (a) $r = 2$; (b) $r = 3$

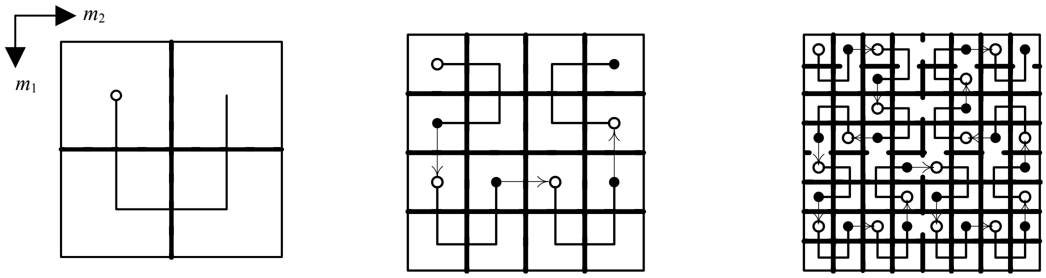
Suppose, $A_0 \in H(\Re)$ denotes a simple curve that connects midpoints of the square subregions (quadrants) $\omega_i(\Re)$ ($i = 1, 2, ..., r^2$) in such a way that each quadrant is entered only once (the traverse direction of the curve being fixed).

Then, the sequence of sets $\{A_n = W^{\circ n}(A_0)\}_{n=0}^{\infty}$, where $W : H(\Re) \to H(\Re)$ is defined by $W(B) = \bigcup_{i=1}^{r^2} \omega_i(B),\ \forall B \in H(\Re)$, converges to the set $\Re$ – attractor of the IFS $\{\Re; \omega_1, \omega_2, ..., \omega_{r^2}\}$ - in the Hausdorff metric. In this case, we say that the sequence $\{A_n\}$ converges to a piecewise continuous space-filling curve, as $n \to \infty$. Each term of the sequence (approximation of the piecewise continuous space-filling curve) A_n ($n \in \{1, 2, ...\}$) comprises $r^{2(n-1)}$ copies of the set (curve) A_0 and is called a piecewise continuous space-filling curve of the n-th degree.

However, this is not quite what we want, because the connecting line segments, indicating the neighbourhood relation (i.e., how the curve leaves and enters the adjacent quadrants), must be introduced. Not going into technical details, we note that, after introduction of the said connecting line segments for each approximation A_n ($n = 1, 2, ...$), the above convergence process results in the so-called (continuous) space-filling curve. Accordingly, each modified (supplemented) approximation A_n ($n \in \{1, 2, ...\}$) is called a (continuous) space-filling curve of the n-th degree.

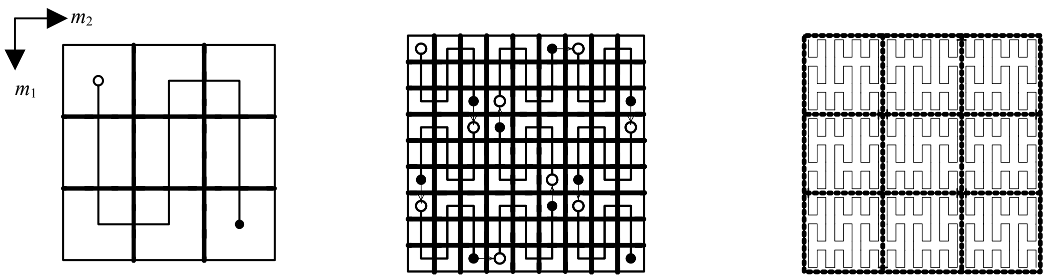
The three-dimensional continuous space-filling curves can be introduced in, practically, the same way.

A few examples of space-filling curves of various degrees are presented in Figures 2 – 4.



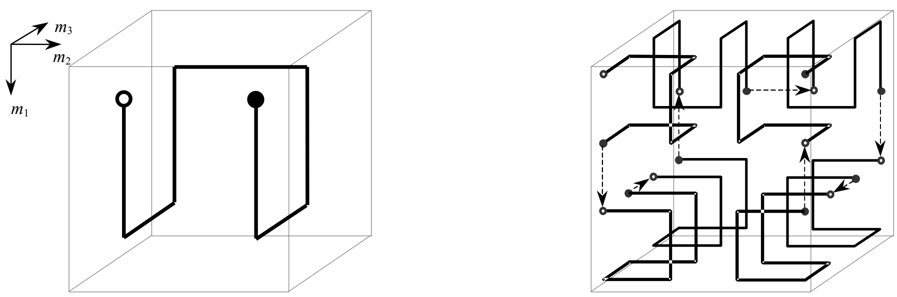
(b) (c)

Figure 2. Hilbert space-filling curves of various degrees (d = 2, r = 2): (a) First degree Hilbert curve; (b) Second degree Hilbert curve; (c) Third degree Hilbert curve



(a) (b) (c)

Figure 3. Peano space-filling curves of various degrees (d = 2, r = 3): (a) First degree Peano curve; (b) Second degree Peano curve; (c) Third degree Peano curve



(a) (b)

Figure 4. Hilbert space-filling curves of various degrees (d = 3, r = 2): (a) First degree Hilbert curve; (b) Second degree Hilbert curve

It goes without saying that the above space-filling curves can be successfully applied to scanning of multi-dimensional digital images, i.e., to generating of sufficiently "good" one-dimensional analogues (data sequences) for them. This intuitive assertion rests on the observation that the scan order, determined by the space-filling curve (Hilbert, Peano, etc.), reflects "geometry" of the image, i.e., nearby pixels in the multi-dimensional image go to nearby pixels in the ordering and vice versa. This fact ensures relatively high level of smoothness of obtainable image representations (one-dimensional image analogues).

To translate the above reasoning into reality, we need to have a means for the generation of successive index values for image elements (pixels), pretending to occupy their positions in one or another ordering (scan). Practically, a few approaches are worthy of attention and can be proposed, namely: the earlier mentioned deterministic fractal synthesis algorithm, Lindermayer systems, specialized procedures, [12, 14, 15]. But, implementation of the first two approaches involves some difficulties, associated with formation of line segments, connecting adjacent quadrants (copies of a piecewise continuous space-filling curve of the first degree). In [15], a specialized recursive image dimensionality change procedure, developed exceptionally for Hilbert space-filling curves (orderings), is presented.

In the section below, a new generalized approach (idea, algorithm) to solving the image dimensionality change problem is proposed. The idea explores both the facts, contained in the first degree space-filling curve ("seed") of arbitrary shape and the relationships between the space-filling curves of the first and second degrees.

4. The image dimensionality change algorithm

Evidently, first degree space-filling curves can be employed to scan a two-dimensional (or, three-dimensional) digital image of size $r \times r$ (or, of size $r \times r \times r$) ($r \in \{2,3\}$; Fig. 2 - 4). To scan an image of arbitrary size $r^n \times r^n$ (or, of size $r^n \times r^n \times r^n$), $n \in \mathrm{N}$, we need to have the two-dimensional (or, three-dimensional) space-filling curve of the n-th degree.

We are to show that the n-th degree space-filling curve (image scan trajectory) can be generated using the first order "seed" matrix $T_0^{(1)}$ and the set of transition matrices $\{Q_0, Q_1, ..., Q_{q-1}\}$, characterizing relationships between the space-filling curves of the first and second degrees (here q stands for the number of image elements (pixels) covered by the "seed"). Matrices of both types are defined and found as soon as the orientation of the "seed" (with respect to coordinate axes in a multi-dimensional space) is fixed (Fig. 2 – 4).

The "seed" matrix of the first order has the form

$$T_0^{(1)} = \begin{pmatrix} T_0^{(1)}(0,0) \ldots\ldots\ldots\ldots T_0^{(1)}(0,d-1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ T_0^{(1)}(q-2,0) \ldots\ldots T_0^{1}(q-2,d-1) \end{pmatrix}, \qquad (5)$$

and its rows specify increments of the (pixel) index values $m_1, ..., m_d$, as we "move" along the "seed" (along the first degree space-filling curve).

For the two-dimensional Hilbert "seed" (Fig. 2, a), we have

$$T_0^{(1)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T, \; q = 4;$$

for the two-dimensional Peano "seed" (Figure 3, a), we have

$$T_0^{(1)} = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T, \; q = 9;$$

for the three-dimensional Hilbert "seed" (Figure 4, a)

$$T_0^{(1)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}^T, \; q = 8.$$

Similar matrices can be constructed for all copies of the "seed", comprising the space-filling curve of the second (or, higher) degree. Their internal structure is identical to that of the matrix $T_0^{(1)}$ (expression (5)). Say, for Hilbert curve of the second degree (Figure 2, b), we obtain four "seed" matrices of the second order:

$$T_0^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^T, \; T_1^{(2)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T,$$

$$T_2^{(2)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T, \; T_3^{(2)} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^T;$$

for Peano curve of the second degree (Fig. 3, b), we have nine "seed" matrices of the second order:

$$T_0^{(2)} = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T,$$

$$T_1^{(2)} = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}^T,$$

....

$$T_8^{(2)} = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T,$$

and, for the spatial Hilbert curve of the second degree (Figure 4, b), eight "seed" matrices of the second order can be constructed:

$$T_0^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{pmatrix}^T,$$

$$T_1^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T,$$

....

$$T_7^{(2)} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{pmatrix}^T.$$

The earlier mentioned transition matrices Q_i ($i = 0, 1, ..., q-1$), characterizing relationships between the space-filling curves of the first and second degrees, or, what is the same, between the "seed" matrices $T_0^{(1)}$ and $T_i^{(2)}$, $i = 0, 1, ..., q-1$, are found by solving the corresponding matrix equations

$$Q_i \cdot T_0^{(1)} = T_i^{(2)},\ i = 0,1,...,\ q-1. \qquad (6)$$

In particular, for the two-dimensional Hilbert curve (Figure 2), transition matrices Q_i $(i = 0,1,2,3)$ take the form:

$$Q_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},\quad Q_1 = Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

As it was mentioned above, to scan a multi-dimensional image $[X(m)] \in S^d(n)$ (i.e., image at the n-th resolution level; $n = \log_r N$, $r \in \{2,3\}$), the use should be made of the d-dimensional space-filling curve of the n-th degree. Since the latter curve comprises $q^{2(n-1)}$ copies (scaled at ratio $1/r^{n-1}$) of the "seed" (the first degree space-filling curve), we need to know all the "seed" matrices of the order n, i.e., matrices $T_i^{(n)}$, $i = 0,1,...,\ q^{2(n-1)}-1$, characterizing the traverse path for all quadrants (N^d/q, in total) of the image $[X(m)]$ (Fig. 5).

We have proved that any "seed" matrix of order n, i.e., $T_i^{(n)}$ $(i \in \{0,1,...,\ q^{2(n-1)}-1\}, n \in \{2,3,...\})$, can be computed using the first order "seed" matrix $T_0^{(1)}$ and a particular subset of the set of transition matrices $\{Q_0, Q_1,..., Q_{q-1}\}$, namely:

$$T_i^{(n)} = Q_{i_0} \cdot Q_{i_1} \cdot ... \cdot Q_{i_{n-2}} \cdot T_0^{(1)}, \qquad (7)$$

where indices $i_0, i_1,..., i_{n-2}$ are found from the q-valued resolution of the decimal number i, i.e.,

$$\begin{aligned} i &= i_{n-1} \cdot q^{n-1} + i_{n-2} \cdot q^{n-2} + ... + i_1 \cdot q + i_0 = \\ &= (i_{n-1}, i_{n-2},..., i_1, i_0) \end{aligned}; \qquad (8)$$

$i_t \in \{0,\ 1,...,\ q-1\}$, for all $t = 0,1,...,n-1$.

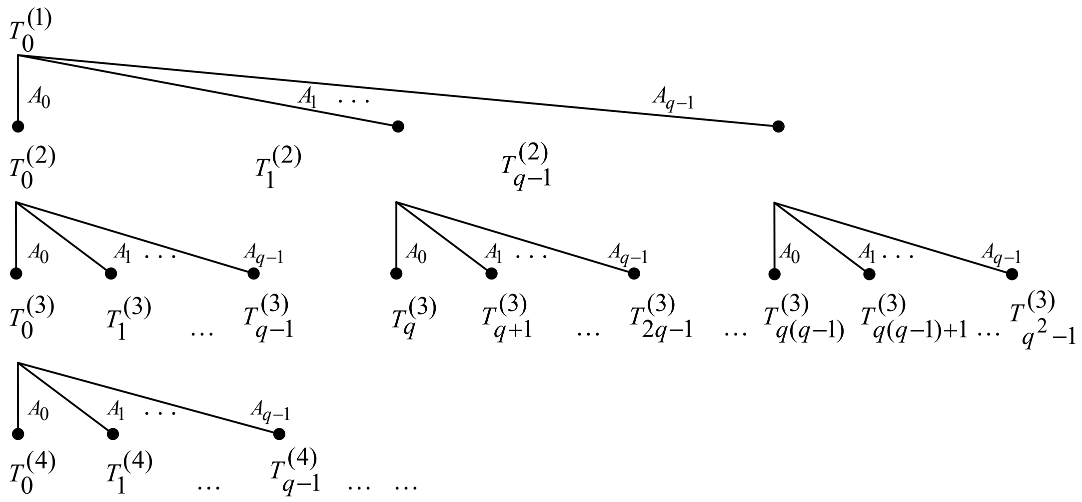


Figure 5.The scheme of relationships between the "seed" matrices of various orders

Based on this understanding, a new generalized image dimensionality change procedure (algorithm) has been developed and is presented below. The multi-dimensional image $[X(m)] \in S^d(n)$ ($n = \log_r N \in \mathrm{N}$, $r \in \{2,3\}$, $d \in \{2,3\}$) is transformed into a one-dimensional data sequence (one-dimensional image) $[V(h)]$, $h = 0,1,...,\ N^d-1$, using the space-filling curve of the n-th degree. Numbering of image elements (pixels), as well as orientation of employed space-filling curves, is chosen in agreement with Figures 2 – 4.

Algorithm /Changing image dimensionality/

1. Specify the first order "seed" matrix $T_0^{(1)}$ and the transmition matrices Q_i $(i = 0,1,...,\ q-1)$ for the space-filling curve of the n-th degree in use; $m = (m_1,..., m_d) := (0,...,0)$, $h := 0$, $[V(h)] := X(m)$, $i := 0$; $h := h+1$.

2. Find the q-valued resolution of the decimal number i (expression 8), i.e.,

$$\begin{aligned} i &= i_{n-1} \cdot q^{n-1} + i_{n-2} \cdot q^{n-2} + ... + i_1 \cdot q + i_0 = \\ &= (i_{n-1}, i_{n-2},..., i_1, i_0) \end{aligned};$$

Take the "seed" matrix $T_i^{(n)}$, which describes the scan trajectory (ordering) of the i-th quadrant of the image $[X(m)]$ under processing, i.e.,

$$T_i^{(n)} = Q_{i_0} \cdot Q_{i_1} \cdot ... \cdot Q_{i_{n-2}} \cdot T_0^{(1)}.$$

3. Scan the i-th quadrant of the image $[X(m)]$, i.e., for all successive values of l ($l = 0, 1,..., q$ -2), do:

$m = (m_1, ..., m_d) :=$
$:= (m_1 + T_i^{(n)}(l,0), ..., m_d + T_i^{(n)}(l, d-1)$;
$[V(h)] := X(m)$; $h := h + 1$.

4. Let t ($t \in \{0,1,...,n-2\}$) be the minimal index such that $i_t < q-1$ (Step 2). Then move to the (i+1)-st (adjacent) quadrant of the image [X(m)] (along the connecting line segment):

$m = (m_1, ..., m_d) :=$
$:= (m_1 + T_{i_{t+1}}^{(n-1-t)}(i_t, 0), ..., m_d + T_{i_{t+1}}^{(n-1-t)}(i_t, d-1)$;
$[V(h)] := X(m)$; $i := i + 1$,

and go to step 2. If $i_t = q-1$, for all $t = 0,1,...,n-2$, then go to step 5.

5. (The end) One-dimensional analogue [V(h)] of the image [X(m)] is generated.

The computational complexity of the proposed image dimensionality change algorithm can be simplified considerably by preliminary evaluation of “seed” matrices of all orders. Since many matrices repeat, we can perform appropriately organized grouping of “seed” matrices and employ them later on efficiently. For instance, there are only four distinct “seed” matrices for the two-dimensional space-filling Hilbert and Peano curves (Figures 2, 3), twelve distinct “seed” matrices for the three-dimensional space-filling Hilbert curve (Figure 4).

5. Task-oriented application of image dimensionality change procedures

One of the most general leading principles, which are at the helm in drawing up digital image processing and analysis strategies, says - the digital image processing should always be performed in the task-oriented image space, which either gives optimum to the objective function (final result), or facilitates the most rational use of a particular specialized image processing algorithm, acting in the chosen image space, [16].

The above principle can be translated into reality if and only if the digital image under processing $[X(m)] \in S^d(n_d)$ ($n_d \in \mathrm{N}, d \in \{1,2,3\}$) allows representations in other (neighbouring) image spaces $(S^{\hat{d}}(n_{\hat{d}}), \delta)$ ($n_{\hat{d}} \in \mathrm{N}$, $\hat{d} \in \{1,2,3\}$, $\hat{d} \neq d$), i.e., if the following condition holds true - $1 \cdot n_1 = 2 \cdot n_2 = 3 \cdot n_3$.

On the other hand, to find those representations (analogues of the image $[X(m)]$), a particular scan trajectory (Hilbert, Peano, scanline ordering, etc.), that gives maximum to the image (analogue) smoothness parameter value in a newly chosen space, should be employed.

Some interesting developments in this field are presented briefly below.

5.1. Hyperbolic image filtering in spaces of different dimensionality

The generalized hyperbolic image filtering idea can be introduced this way. Consider a d-dimensional grey-level digital image $[X(m)] \in S^d(n)$. Let $[Y_X(k)]$ be its d-dimensional discrete spectrum. If, now, M_d is some a priori chosen integer ($1 \le M_d < (N-1)^d$), then for storing one must take only those spectral coefficients $Y_X(k)$, whose serial numbers k (indices $k_1, ..., k_d$) satisfy the condition - $\overline{k_1} \cdot ... \cdot \overline{k_d} \le M_d$ (here $\overline{k_i} = \max\{k_i, 1\}$, i = 1,…, d). When reconstructing the initial image (obtaining its estimate $[\widetilde{X}(m)]$), the rest spectral coefficients (with $\overline{k_1} \cdot ... \cdot \overline{k_d} > M_d$) are equated to zero, i.e., high frequency components of the image at the decompression stage are ignored (compression effect!). Thus, the hyperbolic filtering idea leans upon the supposition that the human eye is less sensitive to changes in high frequencies than in lower ones.

The characteristic features of the hyperbolic image filters – their simplicity, easy realization, tolerable compression ratios (β = 5 – 20, for d = 2, and β = 20 – 100, for d = 3).

When performing hyperbolic filtering in spaces with varying dimensionality, the final result (compression ratio, quality of restored images) strongly depends on the image smoothness class in one or another space. In the publication [2], special (theoretical) criteria, based on the image smoothness analysis results, have been introduced to pick up an optimal space for hyperbolic image filtering. In particular, it has been stated that hyperbolic filtering of the image [X(m)] in the space $S^{d_1}(n_{d_1})$ is more efficient (image compression ratio being fixed!) than in $S^{d_2}(n_{d_2})$, provided

$$(\alpha_{d_1} / \alpha_{d_2}) < (\log M_{d_2} / \log M_{d_1});$$

here: α_{d_1} and α_{d_2} signify smoothness class of the image $[X(m)]$, presented in the image spaces $S^{d_1}(n_{d_1})$ and $S^{d_2}(n_{d_2})$, respectively; M_{d_1} and M_{d_2} stand for the filtering levels in the latter spaces; $d_1, d_2 \in \{1,2,3\}$.

Rather interesting experimental results on this subject were obtained by Mantas Puida (Master of Science in Informatics, Kaunas University of Technology, [17]). For the efficiency analysis of hyperbolic image filters in spaces of different dimensionality, all necessary representations (analogues) of the image under processing were obtained using multi-dimensional Hilbert space-filling curves of a prescribed degree.

In particular, it has been revealed that hyperbolic filtering of two-dimensional images in neighbouring image spaces, softens manifestation of the Gibb's effect (so peculiar to JPEG), resulting from attempting to approximate square wave by a trigonometric polynomial (Figure 6).

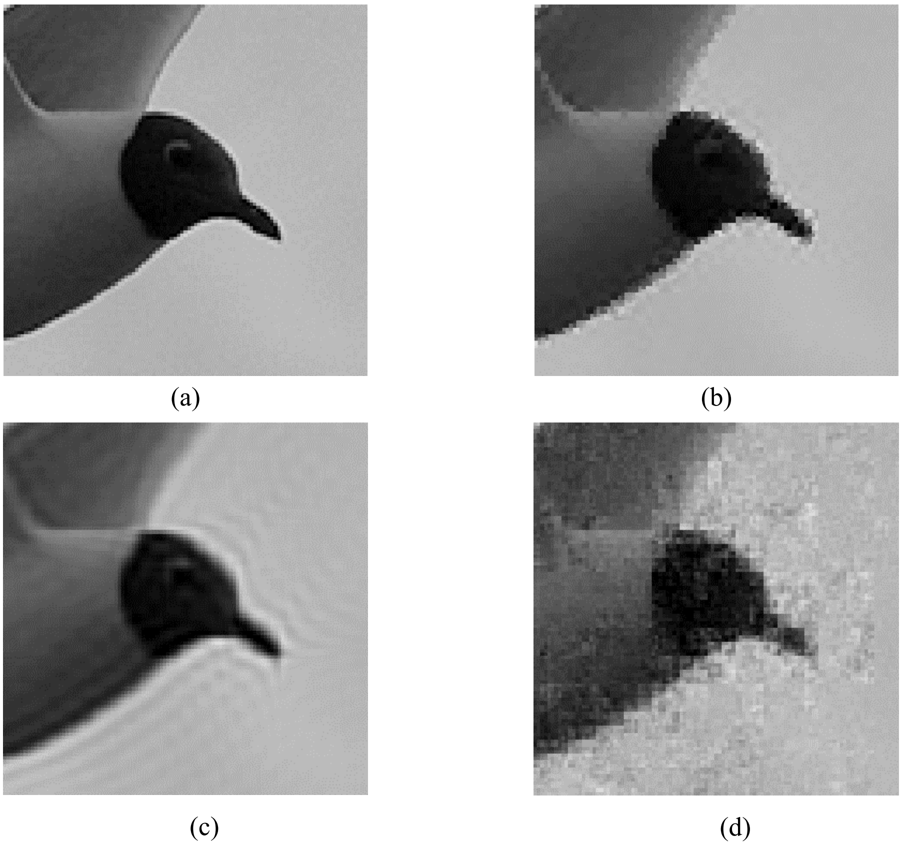


Figure 6. Hyperbolic image filtering in spaces of different dimensionality: (a) original image "Bird" 512x512; smoothness class $\alpha_2 = 0.74$ (DCT); (b) the image estimate, resulting from hyperbolic filtering in the one-dimensional space ($\alpha_1 = 0.57$, $M_1 = 52427$, $\delta = \delta(X, \widetilde{X}) = 4.83$; (c) the image estimate, resulting from hyperbolic filtering in the two-dimensional space ($M_2 = 12959$, $\delta = 4.29$; Gibb's effect is visual); (d) the image estimate, resulting from hyperbolic filtering in the three-dimensional space ($\alpha_3 = 0.75$, $M_3 = 3099$, $\delta = 7.69$);

5.2. Efficient encoding of multi-dimensional binary „silhouette" images

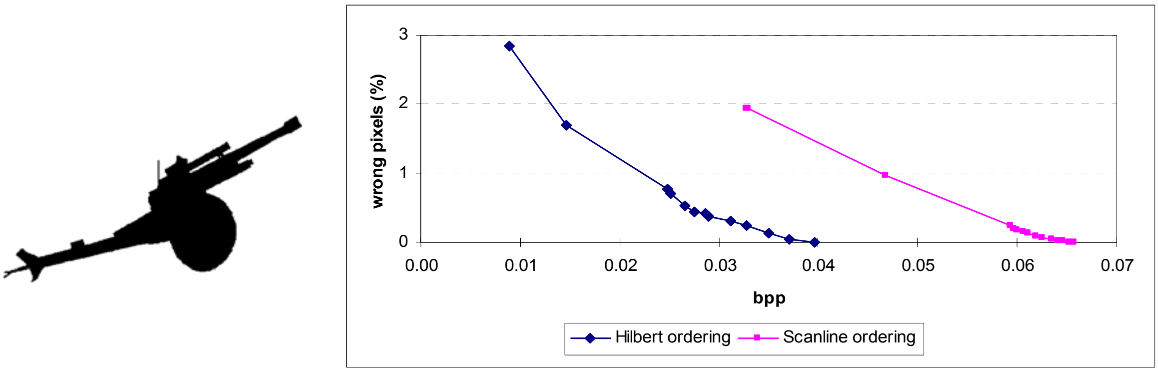
Let $[X(m)] \in S^d(n)$, where $X(m) \in \{0,1\}$, for all $m \in I^d$; $I = \{0,1,...,N-1\}$, $N = r^n$, $r \in \{2,3,...\}$, $d \in \{2,3\}$. Using the image dimensionality change procedure, the image $[X(m)]$ is moved into the image space $(S^1(n \cdot d), \delta)$, i.e. $[X(m)]$ is replaced with its one-dimensional analogue (binary data sequence) $[V(h)]$, $h = 0,1,..., N^d - 1$.

If the initial multi-dimensional binary image $[X(m)]$ is characterized as being "silhouette", then employment of the space-filling curves of the n-th degree (Hilbert, Peano) signifies that the size of monochrome blocks, comprising the generated binary data sequence (one-dimensional image) $[V(h)]$, are expected to be sufficiently large. This perception has gone as the underlying idea throughout the developed "silhouette" image encoding procedure, [15]. The embodied data compression principle rests on the successive elimination of monochrome blocks of maximal size from the image $[V(h)]$. To improve the overall performance of the procedure, the "enforced" enlargement of consecutively removable monochrome blocks is done. The latter circumstance makes the coding procedure lossy.

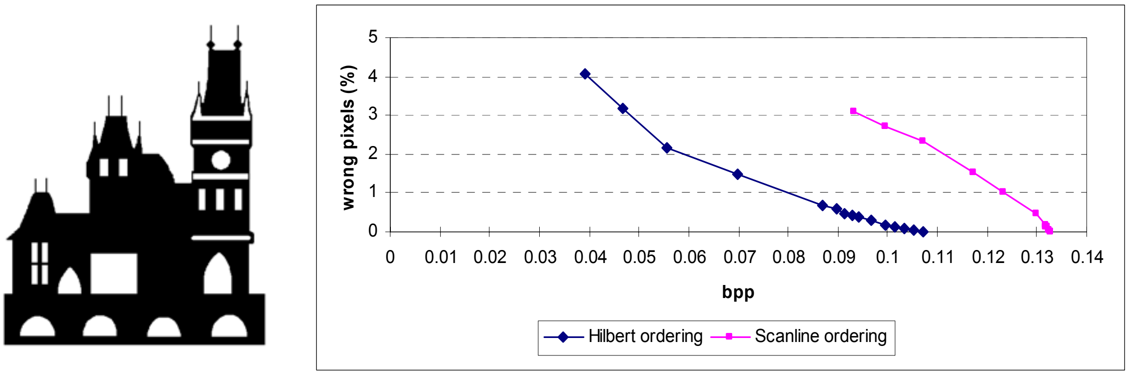
The compressed version of $[V(h)]$ consists of the appropriately made up information, concerning the size, contents and localization of all the eliminated blocks.

Numerous experimental results showed that the image compression ratios achieved highly depend on the shape and localization of monochrome blocks in the "silhouette" image under processing. An obvious advantage of the approach – the presence of lossy encoding (Figure 7).

As it can be seen (Figure 7), application of two-dimensional space-filling Hilbert curves of the ninth degree (in the image dimensionality change process) gave much better results (in terms of "bits-per-pixel"), as compared with scanline ordering (left to right, top to bottom).



(a)



(b)

Figure 7. Efficient encoding of multi-dimensional binary "silhouette" images: (a) Image "Gun.bmp" 512x512 (Lossless coding: Hilbert ordering – 0.039 bpp, scanline ordering – 0.066); (b) Image "Castle.bmp" 512x512 (Lossless coding: Hilbert ordering – 0.107 bpp, scanline ordering – 0.133 bpp)

6. Conclusion

Changing image dimensionality appears to be an interesting, intrigue and perspective (from the practical point of view) idea. It can be successfully employed every time when it is necessary to adapt one or another specialized image-processing technique to images of different dimensionality. The image dimensionality change criteria, in general, may be very specific. But, if image processing is linked with efficient image encoding (compression), the only criterion – preservation of maximal smoothness of the image.

The paper describes a generalized approach (procedure) to changing dimensionality of a multi-dimensional image. The approach employs space-filling curves of various degrees (Hilbert, Peano, etc.), which reflect "geometry" of the image, i.e., nearby pixels in the image go to nearby pixels in the ordering.

Indisputable advantage of the proposed image dimensionality change procedure (approach) – invariance of all the intermediate steps with respect to the geometrical construction and size of a "seed" (space-filling curve of the first degree), i.e., the idea can be easily put into practice, whatever the shape of the "seed" in use.

Some areas of practical application of the developed image dimensionality change procedure are elucidated in the paper, namely: hyperbolic image filtering in spaces of different dimensionality, efficient encoding of multi-dimensional binary "silhouette" images in the one-dimensional image space.

Theoretical and experimental analysis results obtained confirm that the use of task-oriented image dimensionality change procedures forms a new platform for the development, adaptation and productive future analysis of mathematical digital image processing (encoding, filtering, etc.) techniques.

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