ISSN 1392 – 124X INFORMATION TECHNOLOGY AND CONTROL, 2008, Vol. 37, No. 2

MODELING OF THE NEGATIVE SURPLUS IN INSURANCE

Henrikas Pranevicius, Kristina Sutienė*

Department of Business Informatics, Kaunas University of Technology
Studentu 56-301, LT-51424 Kaunas, Lithuania

Abstract. An insurance activity model that allows to evaluate the duration of the negative surplus is presented in this paper. Assuming the surplus process as continuing if ruin occurs, we consider how long this process will stay below zero. The compound Poisson continuous time surplus process is used for the model development. Analytical formulas are obtained for estimating the expected value and the dispersion of both the number and the duration times of negative surpluses when individual claims amount is distributed according to the Gamma (α, β) distribution, with free choice of the parameters $\alpha \in \boldsymbol{N}$ and $\beta > 0$. We use simulation to verify the soundness of the analytical results, evaluating the performance of surplus process under various factors. An aggregate approach has been used for creation of formal description of insurer's business process. The characteristics of the duration of the negative surplus are computed and compared with the theoretical approximate values obtained from the analytical expressions.

Keywords: Non-life insurance, ruin theory, surplus process, aggregate speci.cation, simulation.

1. Introduction

In this paper the insurance activity is described according to the classical risk theory [1], which con-centrates on the claim process, looking first at a claim number, then at a distribution of a claim size and finally putting these two together into an aggregate claims amount process. Premiums are fixed and re-ceived at every time moment. Therefore, the financial operations of an insurer can be viewed in terms of se-ries of cash inflows and outflows. While a lot of work done in calculating ruin probability [2], we are inter-ested in finding some characteristics of how long the surplus will stay below the zero level. Fig. 1 shows a typical sample path of the surplus process consid-ered for this problem.

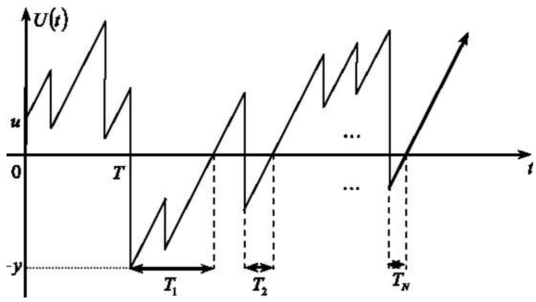

Figure 1. The surplus process

In the figure, T is the time of ruin, u is the initial surplus, y is the severity of ruin, T_1 is the duration of the first negative surplus, T_i ($i>1$) is the duration of any other negative surplus. Let $TT=T_1+T_2++T_N$ represent the total time that surplus process $U(t)$ is below zero, N being the stochastic number of negative surpluses.

The model is based upon earlier work of Reis [3]. In author's paper, the case when ruin occurs is con-sidered. In such situation, the insurance company can ask for a credit to support some negative surplus for some time with the hope that the process will recover in the future. The problem is to consider whether this recovery is quick or not. The author derived the mo-ment generating function of the duration of negative surplus or the time to recovery, given that ruin oc-curs. Since the surplus can fall below the zero level more than once, Reis presented formulas for the dis-tribution of the number of periods, as well as the total duration, of negative surpluses. He demonstrated two examples, considering Exponential and Gamma$(2,\beta)$ individual claims amount distributions, resulting ex-plicit formulas for the target variables. The extension of Reis model is to account for the situation where the individual claims amount is distributed according to the Gamma(α,β), $\alpha,\beta>0$ distribution. This deci-

*Corresponding author. Tel: +370 681 52842 Fax: +370 37 300 352

sion was based on the fact that some kind of insurance loss data typically has a skewed distribution, but does not include extremely high claims which would require the use of long tailed distributions like, for example, the Pareto distribution [4]. Changing the parameters α and β freely, Gamma distribution function may be used to approximate the loss data of such type. The duration of the negative surplus depends on the severity of ruin, which means that the distribution of the severity of ruin should be known. In article of Sutiene [5], following Reis work it was shown how the conditional distribution of severity of ruin can be derived when the special cases of Gamma(α, β) claims amount are considered. In this paper the formula of the conditional distribution function of severity of ruin that can be used for all combinations of parameters $\alpha \in \mathbf{N}$ and $\beta > 0$ is derived. Using of QR approximate method described in the book [6] solves the estimation problem of the adjustment coefficient for finding the polynomial roots that lay on the right half of the complex plane.

An important aspect of any kind of performance modeling study is to validate the model and its implementation to whatever extent is possible. One way to do this is to study a system using more than one model, for example, both simulation model and analytical model. Following this, the characteristics of the duration of negative surplus from the analytical model are compared with the results from detailed simulation. The simulation model is developed using piece-linear aggregates [7], modeling only premiums income and claims outcome, and then reporting about the duration times of the negative surpluses.

The rest of the paper is organized as follows. In Section 2 the basic model that describes insurance company activity is presented. Some assumptions are given according to the purpose of this paper. In Section 3 the distribution of severity of ruin, as well as moments, are derived using Gamma$(\alpha, \beta), \alpha \in \mathbf{N}$, $\beta > 0$ individual claims amount. In Section 4 some features concerning the duration of negative surplus, that is, its moment generating function and respective moments, such as the expected value and the dispersion are presented. In Section 5 the aggregate approach that has been used for the creation of a formal description of insurer's business activity is presented. The following section relates to the analytical model validation performed by using an appropriate simulation model. Investigation of these models is performed, determining the factors that have effect on the duration of period of the negative surplus.

2. Description of the basic model

In this section, a mathematical model for the variations in the amount of an insurer's surplus over an extended period of time is presented. By surplus we will mean the excess of some initial fund plus premiums collected over claims paid. This is a convenient mathematical, but not accounting definition of surplus. Formally, the insurer's surplus process $\{U(t); t \geq 0\}$ at time t can be expressed as

$$U(t) = u + ct - S(t),$$

where $u = U(0)$ is the initial surplus, c is the premium rate per unit time, $\{S(t); t \geq 0\}$ is an aggregate claims process up to time t. It is assumed that aggregate claims are described according to the compound Poisson process with Poisson parameter λ and are given by

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)},$$

where $\{X_i\}_{i=1}^{\infty}$ is a sequence of independently and identically distributed random variables: X_i is the amount of the ith claim and has the density function $p(x)$, the cumulative distribution function $F(x)$ with $F(0) = 0$, the moment generating function $M_X(z)$ for (at least) $z \leq 0$ and the kth moment about the origin $E[X^k]$; $\{N(t); t \geq 0\}$ is a Poisson claims number process with Poisson parameter λ. Fig. 2 illustrates the aggregate claims process graphically; every occurrence which gives rise to a claim is presented by a vertical step, the height of the step indicating the amount X_i of the claim. In the basic model we have that S only changes by means of jumps, which occur at random times, the heights of the successive jumps are not all equal to one, instead being a sequence of independently and identically distributed random variables. Time is measured to the right along the horizontal axis and the altitude $S(t_i)$ of the stepped line at time t_i shows the total amount of claims during the time interval $(0, t_i]$. The expected value and

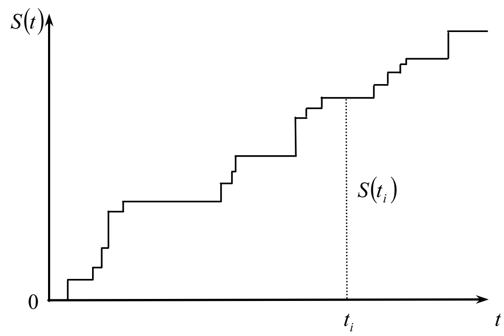

Figure 2. A sample path of claim process

the dispersion of aggregate claims are expressed by $E[S(t)] = E[X]\lambda t$ and $D[S(t)] = E[X^2]\lambda t$, respectively.

In the development of the basic model we consider Gamma(α, β), ($\alpha \in \mathbf{N}$, $\beta > 0$) individual claims amount of the form

$$F_X(x) = 1 - e^{-\beta x} \sum_{k=0}^{\alpha-1} \frac{(\beta x)^k}{k!},$$
$$E(X^k) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}, \qquad k \geq 1, \tag{1}$$

and premium income rate, as follows

$$c = (1+\theta)\lambda\alpha/\beta. \tag{2}$$

With assumption $c > \lambda E(X)$, we claim that per unit time premium income exceeds the expected outgo, i.e. the surplus will go to infinity with probability one, whether or not ruin occurs. Note that the surplus increases linearly except at those times when claims occur. Then the surplus declines by the amount of the claim. As illustrated in Fig. 1, the surplus might become negative at certain times. Considering the duration of this negative surplus we need the conditional distribution function of Y, the severity of ruin. The ruin is said to occur if the insurer's surplus reaches the zero level. It is also necessary to calculate the probability of ruin that depends from both the initial surplus u and the deficit y at time of ruin. The derivation of these probabilities is given in the next section.

3. The distribution of severity of ruin

The time of ruin is denoted as T and defined by

$$T = \begin{cases} \inf(t : U(t) < 0), \\ \infty \quad \text{if } U(t) \geq 0 \text{ for all } t > 0. \end{cases}$$

Let $\psi(u)$ be the probability of ruin with initial surplus u. This probability is given by

$$\psi(u) = \Pr[T < \infty | U(0) = u].$$

The complementary probability is known as the survival probability $\delta(u) = 1 - \psi(u)$. The probability that ruin occurs with the initial surplus u and that the deficit at the time of ruin is less than y is given by

$$G(u,y) = \Pr[U(t) > -y, \forall t : t \geq 0 | U(0) = u],$$

and its density is denoted as $g(u,y)$. Taking the relation $H(u,y) = G(u,y)/\psi(u)$, we will get the conditional distribution function of severity of ruin provided ruin has occurred. Its density function is denoted as $h(u,y)$. In general, it is not possible to find explicit solution for ruin probabilities $\psi(u)$, which results that probabilities $G(u,y)$ and $H(u,y)$ don't have explicit solutions, also. This problem is solved by finding an approximate formula for the probability of ruin, where the adjustment coefficient R plays a key role in the case of light-tailed claims. The value of R can be derived from the following equation

$$e^{-cR} M_{N(t)}(\log M_X(R)) = 1, \tag{3}$$

where $M(z)$ denotes the moment generating function. If there exists a non-zero solution to the Eq. (3), we call such R as an adjustment coefficient. Considering given assumptions of the basic model, Eq. (3) leads to

$$1 + (1+\theta)\frac{\alpha}{\beta}R = \left(\frac{\beta}{\beta - R}\right)^{\alpha}, \tag{4}$$

where $\theta \in (0,1]$ is the insurer's premium loading factor. As noted by Gerber and Shiu [8], in the right half of the complex plane the solutions of Eq. (4) play an important role in calculating ruin probabilities . Performing a change $w = \beta/(\beta - R)$, we get the polynomial

$$\mathrm{w}^{\alpha} + w^{\alpha-1} + \cdots + w - \alpha(1+\theta) = 0, \qquad \alpha \in N, \tag{5}$$

which is of degree α, with $(\alpha+1)$ real number coefficients. In the field of complex numbers, the polynomial in Eq. (5) has at most α roots. If the parameter $\alpha = \overline{1,3}$, we use exact formulas for finding roots w. In case $\alpha > 3$ we construct the following matrix

$$P = \begin{pmatrix} -1 & -1 & \cdots & -1 & \alpha(1+\theta) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where values are taken from polynomial Eq. (5). In such way the problem of finding polynomial roots w is reduced to calculating eigenvalues of the matrix P, using the QR method described by Plukas [6]. Complex roots make a couple, because the polynomial coefficients are real. If α is an even number, we have two real roots w: one of them is negative, another is larger than one. If α is an odd number, we have one real root w, larger than one. That's why we claim that in the right half of the complex plane, the Eq. (4) has α adjustment coefficients. We denote these α roots by $R_1, R_2, \ldots, R_\alpha$. Using the formula taken from Karpickaite [9] and considering Gamma(α, β),

($\alpha \in \mathbf{N}$, $\beta > 0$) individual claims amount, the approximate formula for probability of ruin is

$$\psi(u) = \sum_{k=1}^{\alpha} C_k e^{-R_k u}, \qquad (6)$$

where $C_k = \theta/((\beta/(\beta - R_k))^{\alpha+1} - (1+\theta))$ is constant, $k = \overline{1,\alpha}$. For exponentially distributed claims the Eq. (6) yields an exact result. In other cases it is an approximate formula of $\psi(u)$ as $u \to \infty$. Setting extreme values of the initial surplus $u = 0$ and $u = \infty$, the well known relations follow

$$\psi(0) = \frac{1}{1+\theta}, \qquad \psi(\infty) = 0.$$

In the paper of Karpickaite [10] the exact method was used to derive the analytical formula for the density of ruin probability $g(u, y)$. Employing her result and applying the model assumptions, it follows

$$g(u,y) = \frac{\beta}{\alpha\theta} e^{-\beta y} \sum_{k=1}^{\alpha} C_k \left(\sum_{m=0}^{\alpha-1} \left(-\beta^m + \frac{\beta^\alpha}{(\beta - R_k)^{\alpha-m}} \right) \frac{y^m}{m!} \right) e^{-R_k u}. \qquad (7)$$

The probability function is then given by

$$G(u,y) = \frac{\beta}{\alpha\theta} \sum_{k=1}^{\alpha} C_k \left(\frac{\alpha\theta}{\beta} + e^{-\beta y} \left(-\frac{\alpha\theta}{\beta} + \sum_{s=1}^{\alpha-1} \sum_{m=s}^{\alpha-1} \left(\beta^{s-1} - \frac{\beta^{\alpha-m+s-1}}{(\beta - R_k)^{\alpha-m}} \right) \frac{y^s}{s!} \right) \right). \qquad (8)$$

In this paper we used the agreements that $\sum_{k=1}^{0}(\cdot) = 0$ and $0! = 1$. To simplify Eqs. (7) and (8), we used the following identity derived during the investigation

$$\frac{-\alpha}{\beta} + \sum_{k=1}^{\alpha} \frac{\beta^{k-1}}{(\beta - R)^k} \equiv \frac{\alpha\theta}{\beta}, \qquad \forall R.$$

The formulas (7) and (8) are rather complicated and need to be checked for correctness. It is well-known that

$$g(0,y) = \frac{1}{(1+\theta)E(X)} (1 - F_X(y)),$$
$$G(0,y) = \frac{1}{(1+\theta)E(X)} \int_0^y (1 - F_X(x))\, dx,$$

from which it follows

$$g(0,y) = \frac{\beta}{(1+\theta)\alpha} e^{-\beta y} \sum_{k=0}^{\alpha-1} \frac{(\beta y)^k}{k!}, \qquad (9)$$

$$G(0,y) = \frac{1}{(1+\theta)} - \frac{e^{-\beta y}}{\alpha(1+\theta)} \times \sum_{k=0}^{\alpha-1} \frac{\beta^k}{k!} \left(y^k + \sum_{m=1}^{k} \frac{(-1)^m k(k-1)\cdots(k-m+1)}{(-\beta)^m} \times y^{k-m} \right), \qquad (10)$$

given that $F_X(x) = 1 - e^{-\beta x} \sum_{k=0}^{\alpha-1} \frac{(\beta x)^k}{k!}$ and $E(X) = \frac{\alpha}{\beta}$. Setting $u = 0$ for Eqs. (7) and (8) yields

$$g(0,y) = \frac{\beta}{\alpha\theta} e^{-\beta y} \sum_{k=1}^{\alpha} C_k \left(\sum_{m=0}^{\alpha-1} \left(-\beta^m + \frac{\beta^\alpha}{(\beta - R_k)^{\alpha-m}} \right) \frac{y^m}{m!} \right), \qquad (11)$$

$$G(0,y) = \frac{\beta}{\alpha\theta} \sum_{k=1}^{\alpha} C_k \left(\frac{\alpha\theta}{\beta} + e^{-\beta y} \left(-\frac{\alpha\theta}{\beta} + \sum_{s=1}^{\alpha-1} \sum_{m=s}^{\alpha-1} \left(\beta^{s-1} - \frac{\beta^{\alpha-m+s-1}}{(\beta - R_k)^{\alpha-m}} \right) \frac{y^s}{s!} \right) \right), \qquad (12)$$

which in analytical form can't be compared with Eqs. (9) and (10), respectively. Finding identities

$$\sum_{k=1}^{\alpha} C_k \equiv \frac{1}{1+\theta},$$
$$\sum_{k=1}^{\alpha} \frac{C_k}{(\beta - R_k)^m} \equiv \frac{1}{\beta^m}, \qquad \forall m = 1, 2, \ldots, \alpha,$$

result the same formulas as (9) and (10). Setting $y = \infty$ in Eq. (12) leads to the following

$$G(0,\infty) = \sum_{k=1}^{\alpha} C_k = \frac{1}{1+\theta}.$$

During this procedure we checked the expressions of ruin probabilities under extreme values and obtained well known relations that verify the derived formulas.

Thus, we can derive the conditional distribution function of severity of ruin for Gamma(α,β) individual claims amount that is given by

$$\begin{aligned}H(u,y) = {}& \frac{\beta}{\alpha\theta}\sum_{k=1}^{\alpha} C_k\Biggl(\frac{\alpha\theta}{\beta} + e^{-\beta y}\Biggl(-\frac{\alpha\theta}{\beta} \\ &+ \sum_{s=1}^{\alpha-1}\sum_{m=s}^{\alpha-1}\biggl(\beta^{s-1} \\ &- \frac{\beta^{\alpha-m+s-1}}{(\beta-R_k)^{\alpha-m}}\biggr)\frac{y^s}{s!}\Biggr)\Biggr) \\ &\times e^{-R_k u}\Big/\sum_{k=1}^{\alpha} C_k e^{-R_k u},\end{aligned}$$

and its respective density as

$$\begin{aligned}h(u,y) = {}& \frac{\beta}{\alpha\theta}e^{-\beta y}\sum_{k=1}^{\alpha} C_k\Biggl(\sum_{m=0}^{\alpha-1}\biggl(-\beta^m \\ &+ \frac{\beta^{\alpha}}{(\beta-R_k)^{\alpha-m}}\biggr)\frac{y^m}{m!}\Biggr) \\ &\times e^{-R_k u}\Big/\sum_{k=1}^{\alpha} C_k e^{-R_k u}.\end{aligned}$$

After some simplification, we get the equivalent expressions for the distribution of severity of ruin

$$H(u,y) = \sum_{k=0}^{\alpha-1} A_{\alpha-k}(u)\Biggl(1 - e^{-\beta y}\sum_{m=0}^{k}\frac{(\beta y)^m}{m!}\Biggr), \tag{13}$$

and

$$h(u,y) = e^{-\beta y}\sum_{k=0}^{\alpha-1} A_{\alpha-k}(u)\frac{\beta^{(k+1)}}{k!}y^k, \tag{14}$$

where

$$A_z(u) = \frac{\sum_{k=1}^{\alpha} C_k(-1 + (\frac{\beta}{\beta-R_k})^z)e^{-R_k u}}{\alpha\theta\sum_{k=1}^{\alpha} C_k e^{-R_k u}}, \quad z = \overline{1,\alpha}.$$

The conditional distribution of severity of ruin in Eqs. (13) and (14) can be expressed as a mixture of Gamma$(1,\beta)$, Gamma$(2,\beta), \ldots,$ Gamma(α,β) distributions, with weight functions $A_\alpha(u), A_{\alpha-1}(u), \ldots, A_1(u)$, respectively. When the initial surplus $u = 0$, we get $A_\alpha(0) = A_{\alpha-1}(0) = \cdots = A_1(0) = 1/\alpha$. When u increases, weight functions stabilize very quickly, the "Gamma$(1,\beta)$ part" tends to have more importance and the "Gamma(α,β) part" tends to have less importance. For large values of u the weight function can be expressed as

$$A_z(\infty) = \frac{-1 + (\frac{\beta}{\beta-R})^z}{\alpha\theta}, \qquad z = \overline{1,\alpha},$$

where R denotes the real value of adjustment coefficient in case of α is an odd number. If α is an even number, R is the lesser value of two real adjustment coefficients. Using the distribution mixture Eq. (14), the moment generating function (as well as moments) of severity of ruin can be derived

$$\begin{aligned}M_Y(z;u) &= \sum_{k=0}^{\alpha-1} A_{\alpha-k}(u)\biggl(\frac{\beta}{\beta-z}\biggr)^{k+1}, \quad z<\beta, \\ E[Y|u] &= \frac{1}{\beta}\sum_{k=0}^{\alpha-1}(k+1)A_{\alpha-k}(u), \\ E[Y^2|u] &= \frac{1}{\beta^2}\sum_{k=0}^{\alpha-1}(k+1)(k+2)A_{\alpha-k}(u), \\ D[Y|u] &= \frac{1}{\beta^2}\Biggl(\sum_{k=0}^{\alpha-1}(k+1)(k+2)A_{\alpha-k}(u) \\ &\quad - \Biggl(\sum_{k=0}^{\alpha-1}(k+1)A_{\alpha-k}(u)\Biggr)^2\Biggr).\end{aligned}$$

4. Duration of the negative surplus

The results from previous section are used in deriving formulas for the moments of duration of negative surplus. We will consider the duration of the first negative surplus, duration of any other negative surplus and total duration of negative surpluses. As noted by Reis [3], the moment generating function of the first negative surplus can be considered as a function of the moment generating function of the severity of ruin. Then the moment generating function of T_1 is given by

$$M_{T_1}(f(s);u) = \sum_{k=0}^{\alpha-1} A_{\alpha-k}(u)\biggl(\frac{\beta}{\beta-f(s)}\biggr)^{k+1},$$

where $f(s)$ is some function of s such that

$$s = f(s)c - \lambda\bigl(M_X(f(s)) - 1\bigr).$$

The properties of the function $f(s)$ can be found in Gerber's work [11]. Then, we get the expected value and the dispersion of the first negative surplus

$$E[T_1|u] = \frac{1}{\alpha\lambda\theta}\sum_{k=0}^{\alpha-1}(k+1)A_{\alpha-k}(u),$$

$$\begin{aligned} D[T_1|u] = {} & \frac{(\alpha+1)}{(\alpha\lambda)^2\theta^3}\sum_{k=0}^{\alpha-1}(k+1)A_{\alpha-k}(u) \\ & + \frac{1}{(\alpha\lambda\theta)^2}\left(\sum_{m=0}^{\alpha-1}(m+1)(m+2)\right. \\ & \times A_{\alpha-m}(u) \\ & \left. - \left(\sum_{l=0}^{\alpha-1}(l+1)A_{\alpha-l}(u)\right)^2\right). \end{aligned}$$

Note that these characteristics depend on u, in general. In dealing with other possible durations $T_i, i > 1$ of negative surpluses, Reis notices that there is no need of finding the distribution of the severity of ruin, because it is possible to calculate its moments as functions of the moments of the individual claims amount. Since the initial surplus is depleted during the first negative surplus, we can use the same formulas as in calculating the duration of the first negative surplus except that the initial surplus should be considered equal to zero. We have then the moment generating function of any other negative surplus

$$M_{T_i}(f(s);0) = \frac{1}{\alpha}\sum_{k=0}^{\alpha-1}\left(\frac{\beta}{\beta-f(s)}\right)^{k+1},$$

and the moments

$$E[T_i|u=0] = \frac{(\alpha+1)}{2\alpha\lambda\theta},$$

$$D[T_i|u=0] = \frac{(\alpha+1)(6(\alpha+1)+4\theta(\alpha+2)-3\theta(\alpha+1))}{12(\alpha\lambda)^2\theta^3}.$$

While considering the total duration of the negative surplus, we will discuss the special case of $u = 0$ and the general case of $u \geq 0$. We begin with the random number of negative surpluses. While the initial surplus is considered to be zero, the number N of negative surpluses follows the geometric distribution, with the moment generating function

$$M_N(z; u=0) = \frac{\theta}{1+\theta-e^z}$$

and the moments

$$E[N|u=0] = \frac{1}{\theta},$$

$$D[N|u=0] = \frac{1+\theta}{\theta^2}.$$

In special case the total duration time TT of negative surpluses has a compound geometric distribution. It then follows

$$M_{TT}(s; u=0) = \alpha\theta \Bigg/ \left(\alpha(1+\theta) - \sum_{k=0}^{\alpha-1}\left(\frac{\beta}{\beta-f(s)}\right)^{k+1}\right),$$

$$E[TT|u=0] = \frac{(\alpha+1)}{2\alpha\lambda\theta^2},$$

$$D[TT|u=0] = \frac{(\alpha+1)(9(\alpha+1)+4\theta(\alpha+2))}{\theta^2}.$$

In general case of $u \geq 0$, the moment generating function of the number N of negative surpluses has the following form

$$M_N(z;u) = 1 - \sum_{k=1}^{\alpha} C_k e^{-R_k u} + \frac{\theta e^z \sum_{m=1}^{\alpha} C_k e^{-R_m u}}{1+\theta-e^z},$$

and the expected value and the dispersion of negative surpluses is as follows

$$E[N|u] = \frac{1+\theta}{\theta}\sum_{k=1}^{\alpha} C_k e^{-R_k u},$$

$$\begin{aligned} D[N|u] = {} & (1+\theta)\sum_{k=1}^{\alpha} C_k e^{-R_k u} \\ & \times \left(\left(1 - \sum_{m=1}^{\alpha} C_m e^{-R_m u}\right)\right. \\ & \left. \times (1+\theta) + 1\right) \Big/ \theta^2. \end{aligned}$$

As noted by Reis, in general case the T_i's are not independently and identically distributed, because the first negative surplus occurs with probability $\psi(u)$, $u \geq 0$. Thus, any other negative surplus will occur with probability $\psi(0)$. The moment generating function of total duration time is obtained as

$$\begin{aligned} & M_{TT}(s;u) \\ & = 1 - \alpha\theta\sum_{k=1}^{\alpha} C_k e^{-R_k u} \\ & \times \left(\sum_{m=0}^{\alpha-1} \frac{A_{\alpha-m}(u)}{(\alpha(1+\theta) - \sum_{l=0}^{\alpha-1}(\frac{\beta}{\beta-f(s)})^{l+1})}\right. \\ & \left. \times \left(\frac{\beta}{\beta-f(s)}\right)^{m+1}\right). \end{aligned}$$

The moments of total duration time of negative surpluses are expressed as

$$E[TT|u] = \sum_{k=1}^{\alpha} C_k e^{-R_k u}\left(\frac{1}{\alpha\lambda\theta}\sum_{m=0}^{\alpha-1}(m+1) \times A_{\alpha-m}(u) + \frac{(\alpha+1)}{2\alpha\lambda\theta^2}\right),$$

$$E[TT^2|u] = \sum_{k=1}^{\alpha} C_k e^{-R_k u}\left(\frac{1}{(\alpha\lambda\theta)^2} \times \sum_{m=0}^{\alpha-1}(m+1)(m+2)\cdot A_{\alpha-m}(u) + \frac{2(\alpha+1)}{(\alpha\lambda)^2\theta^3}\sum_{l=0}^{\alpha-1}(l+1)A_{\alpha-l}(u) + \frac{3(\alpha+1)^2+(\alpha+1)(\alpha+2)\theta}{3(\alpha\lambda)^2\theta^4}\right),$$

$$D[TT|u] = E[TT^2|u] - E^2[TT|u].$$

5. Formalization of an insurance activity

In order to create a simulation model, the Piece Linear Aggregates (PLA) approach [7] is used, which permits to develop formal specification of simulated system. Aggregate is the mathematical scheme that belongs to the class of hybrid automation. The state of the aggregate consists of two components: discrete and continuous. In the intervals with no input signals, the continuous coordinates change according to the linear law. Discrete components of state change only due to the input signals or when continuous coordinate becomes zero. Using PLA formalism we will give the formal specification of the surplus process. The goal is to find some characteristics for both the number and the duration of negative surpluses during the defined period.

The structure of the insurer's activity corresponds to the analytical model described above: premiums are added to the insurer's asset, while the wealth is depleted by claim payments. So there are two flows in the system: the flow of premiums and the flow of claims. The discrete version of model is

$$U_t = U_{t-1} + c - \sum_{i=1}^{N_t} X_i, \qquad t \geq 1,$$

where $U_0 = u$ is defined by user. Modeled values are the first negative surplus, the number and the total duration of all negative surpluses.

The simulated system is described by single aggregate. Ten items present the aggregate specification of the modeled activity:

1. The set of input signals $X = \{\}$.
2. The set of output signals $Y = \{\}$.
3. The set of external events $E' = \{\}$.
4. The set of internal events $E'' = \{e''_1, e''_2\}$, where $e''_1 = \{e''_{1j}\}$, $j = \overline{1,\infty}$ – the arrival of premium, $e''_2 = \{e''_{2j}\}$, $j = \overline{1,\infty}$ – the arrival of accumulated claim.
5. The transition rates between the system states: $e''_1 \mapsto \{\xi^1_j\}$, $j = \overline{1,\infty}$ – time period between the jth and the $(j-1)$th premium arrival to the system, $e''_2 \mapsto \{\xi^2_j\}$, $j = \overline{1,\infty}$ – time period between the jth and the $(j-1)$th accumulated claim arrival to the system.
6. $\nu(t_m) = \{n(t_m), T1(t_m), NR(t_m), TT(t_m)\}$ is the discrete component, where $n(t_m)$ – insurer's surplus value, $T1(t_m)$ – duration of the first negative surplus, $NR(t_m)$ – number of negative surpluses, $TT(t_m)$ – total duration of all negative surpluses.
7. $z_v(t_m) = \{w(e''_1, t_m), w(e''_2, t_m)\}$ is the continuous component of the state, where $w(e''_1, t_m) = t_m$ – time moment of the next premium arrival to the system, $w(e''_2, t_m) = t_m$ – time moment of the next accumulated claim arrival to the system.
8. Parameters: $u \geq 0$ – the initial surplus of insurer, $\lambda > 0$ – the Poisson parameter, $\alpha \in \mathbf{N}$, $\beta > 0$ – Gamma distribution parameters, $\theta \in (0, 1]$ – the safety loading.
9. Initial state $\nu(t_0) = \{u, 0, 0, 0\}$, $z_v(t_0) = \{t_0 + \xi^1_1, t_0 + \xi^2_1\}$.
10. Transition operators

$H(e''_1)$:

$$n(t_{m+1}) = n(t_m) + (1+\theta)\lambda\alpha/\beta,$$

$$w(e''_1, t_{m+1}) = t_m + \xi^1_j,$$

$$\begin{cases} NR(t_{m+1}) = NR(t_m) + 1, & \text{if } n(t_{m+1}) < 0 \\ & \wedge n(t_m) \geq 0 \\ NR(t_{m+1}) = NR(t_m), & \text{otherwise} \end{cases}$$

$$\begin{cases} TT(t_{m+1}) = TT(t_m) + 1, & \text{if } n(t_{m+1}) < 0 \\ TT(t_{m+1}) = TT(t_m), & \text{otherwise} \end{cases}$$

$$\begin{cases} T1(t_{m+1}) = T1(t_m) + 1, & \\ \quad \text{if } n(t_{m+1}) < 0 \wedge NR(t_{m+1}) = 1 & \\ T1(t_{m+1}) = T1(t_m), & \text{otherwise} \end{cases}$$

$H(e''_2)$:

Generate $k \sim$ Poisson(λ)
if $k = 0$ then $n(t_{m+1}) = n(t_m)$
else for $i = 1$ to k
generate $X_i \sim$ Gamma(α, β)
$n(t_{m+1}) = n(t_m) - X_i$
end if
$w(e''_2, t_{m+1}) = t_m + \xi^2_j$.

Table 1. Model parameters that form the base case scenario

Parameter	Value	Meaning
θ	0.2	Safety loading
u	1	Initial insurer's surplus
α	10	Shape parameter of Gamma distribution
β	2	Scale parameter of Gamma distribution
λ	0.1	Average claims frequency (Poisson parameter)

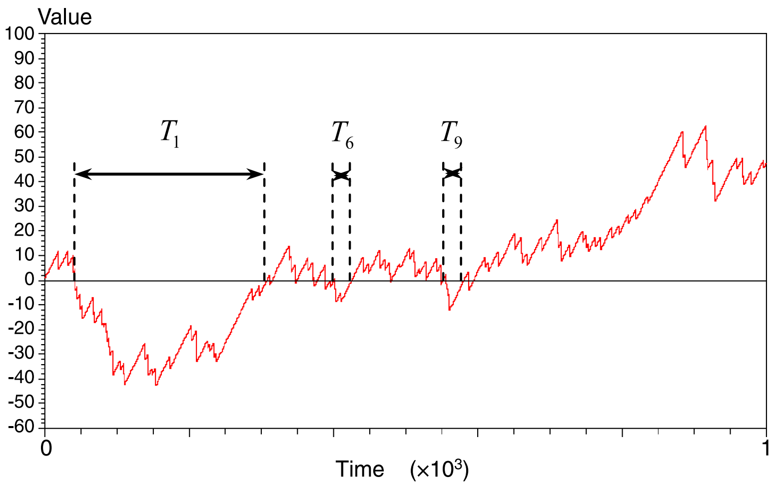

Figure 3. The realization of the modeled surplus process

The formal specification presented above is used for creation of the simulation model using the ARENA simulation modeling and analysis software [12], capturing the behavior of the basic model. The created simulation model is formulated to test the analytical expressions of the duration of the negative surplus under given assumptions.

6. Modeling Results

In the investigation of both analytical and simulation models, the deterministic scenario is used. It's a run of the model listing the particular variables and their particular values. Such a type of scenario is useful at answering "what if" questions. Therefore, it is valuable to have one base case scenario against which to compare the alternative scenarios. Modifying one variable while leaving all others constant can isolate the effect of that variable. Therefore, we start with a base case scenario that serves as a point of reference. Table 1 lists all of the parameters used to obtain the result from the analytical approach, as well as by simulation. Their values are chosen for a basis of model investigation. In the following, the model is simulated over a horizon of ten years simulation time. One possible situation under the base case scenario is illustrated in Fig. 3 : T_1 is the duration of the first negative surplus, T_6 – the duration of the sixth negative surplus, T_9 – the duration of the ninth negative surplus. Then, a set of scenarios, see Table 2, is selected to investigate, altering the model to reflect these scenarios. In defining the factors that give weight to the duration of negative surplus, the chosen parameter is enlarged for three times, and the others parameters are fixed, resulting as an alternative scenario.

By simulating the replications according to the idea of the Monte Carlo method, we estimate the duration of the first negative surplus, both the number and the total duration of negative surpluses. Thus, the simulation consisted of one run of 1000 iterations for each scenario, resulting the confidence interval of required characteristics, meaning that in about 95% of the cases of making one thousand simulation replications as we did, the interval formed like this would "cover" the true expected value of target variable.

As seen from Tables 3 - 5, in alternative 1 scenario the number and the duration of negative surpluses are reduced when bigger value of safety-loading is chosen. While the loading θ has no effect on claim frequency and amount, it determines a steep growth of the process curve yielding the lesser values of duration of negative surpluses. Setting the safety-loading equal to its maximum value, both the duration and the number of the negative surpluses become close to zero. Considering the alternative 2 scenario,

Table 2. The set of alternative scenarios

Model parameters	Alternative 1 scenario	Alternative 2 scenario	Alternative 3 scenario	Alternative 4 scenario	Alternative 5 scenario
θ	0.6	0.2	0.2	0.2	0.2
u	1	3	1	1	1
α	10	10	30	10	10
β	2	2	2	6	2
λ	0.1	0.1	0.1	0.1	0.3

Table 3. Duration T_1 of the first negative surplus.

	Expected value	Dispersion	Confidence interval (95%)
Base case scenario	23.52	6744.04	23.99 $\pm$ 5.72
Alternative 1 scenario	6.81	169.15	5.01 $\pm$ 0.91
Alternative 2 scenario	19.41	5570.77	19.43 $\pm$ 5.07
Alternative 3 scenario	24.27	6495.70	24.78 $\pm$ 6.07
Alternative 4 scenario	19.41	5570.77	19.43 $\pm$ 5.07
Alternative 5 scenario	7.84	749.34	8.91 $\pm$ 1.67

Table 4. Number N of the negative surpluses.

	Expected value	Dispersion	Confidence interval (95%)
Base case scenario	4.82	29.79	4.36 $\pm$ 0.32
Alternative 1 scenario	1.53	4.29	1.30 $\pm$ 0.11
Alternative 2 scenario	4.36	28.95	3.97 $\pm$ 0.33
Alternative 3 scenario	4.94	29.94	4.59 $\pm$ 0.32
Alternative 4 scenario	4.36	28.95	3.97 $\pm$ 0.33
Alternative 5 scenario	4.82	29.79	4.23 $\pm$ 0.23

Table 5. Total duration TT of the negative surpluses.

	Expected value	Dispersion	Confidence interval (95%)
Base case scenario	129.31	59483.78	128.01 $\pm$ 14.75
Alternative 1 scenario	13.28	804.84	12.96 $\pm$ 1.79
Alternative 2 scenario	113.98	54136.92	112.33 $\pm$ 13.94
Alternative 3 scenario	126.40	53782.41	125.31 $\pm$ 14.15
Alternative 4 scenario	113.98	54136.92	112.33 $\pm$ 13.94
Alternative 5 scenario	43.10	6609.31	39.91 $\pm$ 4.63

the value of initial surplus u is enlarged for three times, keeping the other parameters not changed as in base case scenario. In the result the number and the total duration of negative surpluses are reduced. It means that as we increase the initial reserves, we increase the distance between the process values and the critical zero level, not changing the process curve itself. The changing of parameters for Gamma(α, β) distribution results the alternative 3 and alternative 4 scenarios. Choosing the parameter α larger, the individual claims amount, as well as premium, also grows. But varying values of the parameter α has a very small effect on the expected value of the duration of negative surpluses. When the bigger value of Gamma parameter β is chosen, the duration of negative surplus is reduced. For higher values of β the duration of negative surplus of the modeled insurance company becomes equal to zero. Setting the Poisson parameter λ=0.3 yields larger claim frequency, as well as amount of premium. The results of an alternative 5 scenario show that while the duration is reduced, the parameter λ has no effect on the number of negative surpluses.

7. Concluding remarks

The focus of this work was to present an analytical model of the surplus process. The created model can guide in the determination of the duration of negative surplus and allows the investigation of the relationship between the duration and the parameters involved in the model. The obtained analytical results, which were tested by simulation, demonstrate that:

• The moments of both the duration and the number of negative surpluses are very much influenced by the safety loading, especially the dispersion, which is high for low values of initial reserves and safety loading. The dispersion is very much reduced if higher values of these parameters are considered.

- The higher claim frequency λ determines more premium income. That's why it helps to reduce the total duration of negative surpluses of the modeled insurance company, keeping the number of negative surpluses unchanged.
- While the shape parameter α of Gamma distribution has no clear effect on the expected duration of negative surpluses, the scale parameter β of Gamma distribution operates as reduction in the values of duration moments of negative surpluses if higher β is considered.

References

[1] **C.D. Daykin, T. Pentikainen, M. Pesonen.** Practical Risk Theory for Actuaries. *Chapman & Hall, England,* 1996.

[2] **S. Asmussen.** Ruin Probabilities. *Advanced Series on Statistical Science and Applied Probability* 2. W*orld Scienti.c Publishing Co., NY*, 2000.

[3] **A.E. dos Reis.** How Long is the Surplus Below Zero? *Insurance: Mathematics and Economics* (12). *North-Holland,* 1993, 23–28.

[4] **T. Mikosch.** Non-Life Insurance Mathematics: an Introduction with Stochastic Processes. *Springer-Verlag, NY*, 2004.

[5] **K. Sutiene, V. Karpickaite.** Estimation of Average and Dispersion of the Insolvency Duration in Insurance. *Proceedings of Mathematics and Mathemat-ical Modeling Conference, Kaunas*, 2004, 69–74 (*in Lithuanian*).

[6] **K. Plukas.** Numerical Methods and Algorithms. *Naujasis lankas, Kaunas,* 2001, (*in Lithuanian*).

[7] **H. Pranevicius.** Formal Speci.cation and Analysis of Computer Network Protocols: Aggregate Approach. *Technologija, Kaunas*, 2004 (*in Lithuanian*).

[8] **H.U. Gerber, E.S.W. Shiu.** The Time Value of Ruin in a Sparre Andersen Model. *North American Actuarial Journal* (9), 2005, 49–69.

[9] **V. Karpickaite.** Computation of Ruin Probability. *Proceedings of Mathematics and Mathematical Modeling Conference* (2), *Kaunas*, 1998, 108–111 (*in Lithuanian*).

[10] **V. Karpickaite.** Computation of Ruin Probability from Initial Surplus and De.cit in Insurance. *Proceedings of the Mathematics and Mathematical Modeling Conference, Kaunas*, 1999, 24–27 (*in Lithuanian*).

[11] **H.U. Gerber.** When does the Surplus Reach a Given Target? *Insurance. Mathematics and Economics* (9), 1990, 115–119.

[12] **W.D. Kelton, R.P. Sadowski, D.A. Sadowski.** Simulation with Arena. *WCB/McGraw-Hill, USA*, 1998.

Received March 2008.