ISSN 1392 – 124X INFORMATION TECHNOLOGY AND CONTROL, 2008, Vol.37, No.1

ON THE GRAPH COLORING POLYTOPE

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Abstract. The graph coloring problem consists in assigning colors to the vertices of a given graph G such that no two adjacent vertices receive the same color and the number of used colors is as small as possible. In this paper, we investigate the graph coloring polytope $P(G)$ defined as the convex hull of feasible solutions to the binary programming formulation of the problem. We remark that $P(G)$ coincides with the stable set polytope of a graph constructed from the complement $\bar{G}$ of G. We derive facet-defining inequalities for $P(G)$ from independent sets, odd holes, odd anti-holes and odd wheels in $\bar{G}$.

Key words: polyhedral combinatorics, graph coloring, polytopes, facets.

1. Introduction

Given a simple graph $G = (V, E)$ with vertex set V and edge set E, a vertex coloring is an assignment of colors to the vertices so that no two adjacent vertices receive the same color. The graph coloring problem is to find a vertex coloring with the number of used colors as small as possible. This minimum number $\chi(G)$ of colors is called the *chromatic number* of the graph G.

There exist many approaches for the graph coloring problem. These approaches include both exact [2, 4, 10, 16] and heuristic [1, 7, 8, 9] solution methods. In the development of algorithms for graph coloring, various integer programming formulations of the problem could be used. Several such formulations, each involving binary variables, have been proposed: independent set formulation [10], an integer program with a variable for each possible color and vertex [5, 11], a model relating acyclic orientations of a graph to its chromatic number [6], a model with vertices representing colors [3], and a formulation based on star partitioning of the complement of a given graph [14].

Let $\bar{G} = (V, \bar{E})$ denote the complement of the graph $G = (V, E)$ of order $n = |V|$. Assume (which is not restrictive) that each connected component of $\bar{G}$ has order greater than two. Letting V be a set of integers treated as unique identifiers assigned to the vertices of G, we define $T = \{(i, j, k) \mid i < \min\{j, k\}$ and $\{i, j, k\}$ forms a triangle in $\bar{G}\}$ and $\Pi = \{(i, j, k) \mid (i, j), (j, k) \in \bar{E}, (i, k) \notin \bar{E}\}$. The formulation proposed in [14] is as follows:

$$\chi(G) = \min \, (n - \sum_{(i,j)\in\bar{E}} x_{ij}) \tag{1}$$

$$\text{s.t.} \quad x_{ij} + x_{jk} \leqslant 1 \quad \text{for all} \quad (i, j, k) \in T \cup \Pi \tag{2}$$

$$x_{ij} \in \{0, 1\} \quad \text{for all} \quad (i, j) \in \bar{E}. \tag{3}$$

In [14], it is shown that

$$\chi(G) + \alpha(H_G) = n, \tag{4}$$

where $\alpha(H_G)$ is the independence number of the graph $H_G = (V_H, E_H)$ with vertices $t_{ij} \in V_H$ corresponding to $(i, j) \in \bar{E}$ and edges $(t_{ij}, t_{jk}) \in E_H$ corresponding to $(i, j), (j, k) \in \bar{E}$ such that either $(i, j, k) \in \Pi$ or $(i, j, k) \in T$. From equation (4), it is evident that, using (1)–(3), the graph coloring problem can be reduced to that of finding a maximum independent set in a graph. In the current paper, we exploit this relationship between these two problems.

Given a graph G, we define the polytope $P(G) = \text{conv}\,\{x = (x_{ij}), (i, j) \in \bar{E} \mid x$ satisfies (2) and (3) $\}$. When dealing with polytopes, one of the main problems is to identify large classes of inequalities that are facet-defining for them. In the next section, we study the facetial structure of $P(G)$ and derive such inequalities from independent sets, odd holes, odd anti-holes and odd wheels in $\bar{G}$.

We end the introduction with a few basic definitions and notations. A *star* is a tree S with vertex set $V = \{v_1, \ldots, v_l\}$, $l \geqslant 1$, and edge set $E = \{(v_1, v_i) \mid i = 2, \ldots, l\}$ if $l > 1$ and $E = \emptyset$ if $l = 1$. The vertex v_1 is called a *centre vertex* of the star and is denoted by $c(S)$. Given a graph $G = (V, E)$, we denote by $N_i(G)$, $i \in V$ (or just N_i) the set of vertices adjacent to i. The subgraph of G induced by a vertex set $V' \subset V$ is denoted by $G(V')$. The *stable set polytope* $P_{\text{stable}}(G)$ of a graph G is the convex hull of the incidence vectors of the independent (stable) sets in G. The basic concepts of polyhedral theory can be found, for example, in [15].

2. Facets of the polytope

In this section, we exhibit a few classes of facet-defining inequalities for $P(G)$. We start by pointing out a one-to-one correspondence between colorings of G and admissible star partitions of $\bar{G}$. By a *star partition* we understand a collection s of stars $S_i = (V_i, E_i)$, $i = 1, \ldots, l$, in $\bar{G}$ such that $V_i \cap V_j = \emptyset$ for each pair $i, j, i \neq j$, $\cup_{i=1}^{l} V_i = V$, and V_i for each $i \in \{1, \ldots, l\}$ induces a clique in $\bar{G}$. We will write $E(s) = \cup_{i=1}^{l} E_i$. We say that a star partition s is *admissible* if, for each $S_i \in s$, the centre vertex $c(S_i) = \min_{j \in V_i} j$. We denote the set of all admissible star partitions of $\bar{G}$ by $\Psi(\bar{G})$. For $s \in \Psi(\bar{G})$, let $x(s) = (x_{ij}(s) \mid (i,j) \in \bar{E})$ be the incidence vector of s, that is, $0-1$ vector with $x_{ij}(s) = 1$ if and only if $(i,j) \in E_k$ for some star S_k in s. In [14], it is proved that the incidence vectors of star partitions in $\Psi(\bar{G})$ are the only integer points in $P(G)$.

In the rest of this section, we use the following fact, which, in particular, implies that $P(G)$ is full-dimensional (dim$P(G) = |\bar{E}|$).

Proposition 1. $P(G) = P_{\text{stable}}(H_G)$.

The above equation easily follows from (1)–(4) and the definition of $P(G)$. The next assertion is obvious as well.

Proposition 2. For each $(i,j) \in \bar{E}$, the inequality $x_{ij} \geqslant 0$ defines a facet of $P(G)$.

In order to present the first class of nontrivial facet-defining inequalities we need some additional notations. Let $\bar{E}_i$, $i \in V$, denote the set of edges of $\bar{G}$ incident to i. Define $\bar{E}'_i = \{(i,j) \in \bar{E}_i \mid j < i\}$, $\bar{E}''_i = \bar{E}_i \setminus \bar{E}'_i$, $N''_i = \{j \in N_i \mid (i,j) \in \bar{E}''_i\}$. Let I be an inclusion-wise maximal independent set in the graph $\bar{G}(N''_i)$ and $E_i(I) = \{(i,j) \in \bar{E}''_i \mid j \in I\}$. We are interested in the following inequality

$$\sum_{(i,j) \in \bar{E}'_i \cup E_i(I)} x_{ij} \leqslant 1. \tag{5}$$

Theorem 1. For a non-isolated vertex $i \in V$ and any maximal independent set I in the graph $\bar{G}(N''_i)$, the inequality (5) is valid for $P(G)$. In the case where $\bar{G}$ has no connected component of order two, (5) defines a facet of $P(G)$ if and only if $|\bar{E}'_i \cup E_i(I)| \geqslant 2$.

Proof. It is easy to see that the edges of $\bar{E}'_i \cup E_i(I)$ define a clique K in the graph H_G. Therefore, (5) is valid for $P(G)$. Consider the case where $|\bar{E}'_i \cup E_i(I)| \geqslant 2$. Assume that K is not maximal. This means that there exists a vertex $t_{ik} \in V_H$ adjacent to all vertices in K. Clearly, $(i,k) \in \bar{E}''_i$. The maximality of I implies the existence of a vertex $j \in I$ such that $(j,k) \in \bar{E}$. However, $(i,j,k) \in T$ and hence $(t_{ij}, t_{ik}) \notin E_H$, a contradiction to the above assumption. As proved in [13], the inequality $\sum_{i \in V'} x_i \leqslant 1$ for the vertex set V' of an inclusion-wise maximal clique in a graph G defines a facet of $P_{\text{stable}}(G)$. Applying this result to H_G and using Proposition 1, we conclude that (5) is facet-defining for $P(G)$.

If $\bar{E}'_i \cup E_i(I) = \{(i,j)\}$, then, since i belongs to a component of order greater than two, it follows that t_{ij} in H_G is adjacent to at least one other vertex and, therefore, (5) does not define a facet of $P(G)$. □

Notice that (5) can be viewed as a generalization of (2). Indeed, the latter coincides with (5) and thus is facet-defining for $P(G)$ only in the most simple cases. More specifically, (2) for $(i,j,k) \in T$ defines a facet of $P(G)$ if and only if either $N_j = \{i,k\}$ or $\bar{E}'_j = \{(j,i)\}$ and $(k,l) \in \bar{E}$ for each $l \in N''_j \setminus \{k\}$. For $(i,j,k) \in \Pi$, (2) defines a facet of $P(G)$ if and only if either the same condition as for $(i,j,k) \in T$ is satisfied or $\bar{E}'_j = \emptyset$ and $\{i,k\}$ is a maximal independent set in $\bar{G}(N_j)$.

We will now display three classes of inequalities for $P(G)$ with the right-hand side coefficient greater than one. Perhaps, the most simple such inequalities are derived from *odd holes*, that is, graphs $C = (V_C, E_C)$ having vertex set $V_C = \{v_1, \ldots, v_h\}$ of odd cardinality $h \geqslant 5$ and edge set $E_C = \{(v_i, v_{i+1}) \mid i = 1, \ldots, h-1\} \cup \{(v_1, v_h)\}$.

Theorem 2. For an odd hole C in $\bar{G}$, the inequality

$$\sum_{(i,j) \in E_C} x_{ij} \leqslant (h-1)/2 \tag{6}$$

defines a facet of $P(G)$.

Proof. The vertices $t_{v_1 v_h}, t_{v_i v_{i+1}}$, $i = 1, \ldots, h-1$, induce an odd hole $C' = (V_{C'}, E_{C'})$ in the graph H_G. It is well-known [13] that the odd hole inequality

$$\sum_{t_{ij} \in V_{C'}} x_{t_{ij}} \leqslant (h-1)/2 \tag{7}$$

defines a facet of the stable set polytope $P_{\text{stable}}(C')$. It remains to show that this inequality is facet-defining for the polytope $P_{\text{stable}}(H_G)$ too. Let (v_i, v_j) be any edge of $\bar{E} \setminus E_C$. Suppose that $v_i \in V_C$ and $v_j \notin V_C$. Assume for simplicity that $i \in \{2, \ldots, h-1\}$. Then the vertex $t_{v_i v_j}$ is adjacent to at most two vertices of C', namely, $t_{v_{i-1} v_i}$ and $t_{v_i v_{i+1}}$. We form an independent set U by taking $(h-1)/2$ vertices on the path obtained by removing vertices $t_{v_{i-1} v_i}$ and

$t_{v_i v_{i+1}}$ from C'. If $v_i, v_j \notin V_C$, then U is any independent set of size $(h-1)/2$ in C'. The incidence vector of the set $U \cup \{t_{v_i v_j}\}$ satisfies (7) with equality. Considering each $(v_i, v_j) \in \bar{E} \setminus E_C$, we obtain a collection of such vectors, which, obviously, are linearly independent. Consequently, (7) defines a facet of $P_{\text{stable}}(H_G)$. Hence, due to Proposition 1, (6) defines a facet of $P(G)$. □

Let $W = (V_W, E_W)$ denote an *odd wheel* – a graph consisting of an odd hole, called the *rim*, and a vertex connected to all vertices of the rim, called the *hub*. The latter is denoted by c. The complement of an odd hole is called an *odd anti-hole*.

Theorem 3. For an odd wheel W in $\bar{G}$ with $c = \min_{v \in V_W} v$, the inequality

$$\sum_{i \in N_c(W)} x_{ci} \leqslant 2 \tag{8}$$

is valid for $P(G)$. Furthermore, it is facet-defining if the following two conditions are satisfied:

(a) $\bar{E}'_c = \emptyset$, that is, $i > c$ for each $(i, c) \in \bar{E}_c$;

(b) for each $i \notin V_W$ adjacent to c, there exists a pair of vertices $j, k \in N_c(W)$ such that $(j, k) \in E_W$ and $(i, j), (i, k) \in \bar{E}$.

Proof. Since, for an edge (i, j) of the rim, i, j and c induce a triangle, it follows that the vertices t_{ci} and t_{cj} are not adjacent in H_G. On the other hand, the vertex t_{ci}, $i \in N_c(W)$, is connected by an edge to each vertex t_{ck}, $k \in N_c(W)$, $(i, k) \notin E_W$. Therefore, the subgraph of H_G induced by the vertices t_{ci}, $i \in N_c(W)$, is an odd anti-hole $A = (V_A, E_A)$. For its vertex set V_A, the inequality

$$\sum_{t_{ci} \in V_A} x_{t_{ci}} \leqslant 2 \tag{9}$$

defines a facet of the stable set polytope $P_{\text{stable}}(A)$ [12]. This fact establishes the validity of (8).

Now suppose that the stated conditions hold for W. Similarly to the proof of Theorem 2, we construct the required collection of linearly independent incidence vectors. If (i, j) is an edge of the rim, we include in this collection the incidence vector of the independent set $U = \{t_{ij}, t_{ck}, t_{cl}\}$, where $\{k, l\} \cap \{i, j\} = \emptyset$, $k, l \in N_c(W)$ and $(k, l) \in E_W$. If $(i, c) \in \bar{E}$, $i \notin V_W$, then the conditions (a) and (b) allow us to argue that the set $U = \{t_{ci}, t_{cj}, t_{ck}\}$ is independent in H_G and we can use it to represent (i, c); here the meaning of j and k is as in (b). Finally, if $(i, j) \in \bar{E}$, $i \notin V_W$, $j \neq c$, then we can take the set $U = \{t_{ij}, t_{ck}, t_{cl}\}$, where k and l are adjacent vertices of the rim and both k and l differ from j if $j \in N_c(W)$. Since the incidence vectors of the selected sets are linearly independent, it follows that (9) defines a facet of the polytope $P_{\text{stable}}(H_G)$. By virtue of Proposition 1, (8) is facet-defining for $P(G)$. □

A result similar to the above theorems can also be stated for the odd anti-hole $A = (V_A, E_A)$, $|V_A| = h \geqslant 5$.

Theorem 4. For an odd anti-hole A in $\bar{G}$, the inequality

$$\sum_{(i,j) \in E_A} x_{ij} \leqslant h - 3 \tag{10}$$

defines a facet of $P(G)$.

Proof. The validity of (10) for $P(G)$ is obvious. We can assume that $h \geqslant 7$ because if $h = 5$, then A coincides with its complement – the odd hole C. Let $a^T x \leqslant a_0$ denote the inequality (10) and let $b^T x \leqslant b_0$ be a facet-defining inequality for $P(G)$ such that $F_a := \{x \in P(G) \mid a^T x = a_0\} \subseteq F_b := \{x \in P(G) \mid b^T x = b_0\}$. For $u \in V_A$, let $u, v_1, w_1, v_2, w_2, \ldots, v_q, w_q$, $q = (h-1)/2$, be the list of vertices of A ordered in such a manner that (u, v_1), (u, w_q), (v_i, w_i), $i = 1, \ldots, q$, (w_i, v_{i+1}), $i = 1, \ldots, q-1$, are the edges of C (or, equivalently, non-edges of A). We consider cliques in A defined by the vertex sets of the form $K_r = \{u\} \cup \{w_i \mid i = 1, \ldots, r-1\} \cup \{v_i \mid i = r+1, \ldots, q\}$, where $r \in \{1, \ldots, q\}$. We define $\bar{K}'_r = V_A \setminus (K_r \cup \{v_r\})$, $\bar{K}''_r = V_A \setminus (K_r \cup \{w_r\})$. We denote by $S(K_r) = (K_r, E(K_r))$, $r \in \{1, \ldots, q\}$ (similarly, $S(\bar{K}'_r)$, $S(\bar{K}''_r)$) the star with the vertex set K_r and centre vertex $c_r = \min_{v \in K_r} v$. Suppose $(v_r, j) \in \bar{E} \setminus E_A$ and $v_r \in V_A$. For star partition s with $E(s) = E(K_r) \cup E(\bar{K}'_r)$ and star partition s' with $E(s') = E(s) \cup \{(v_r, j)\}$, we have that $x(s), x(s') \in F_a$ and hence $x(s), x(s') \in F_b$ implying $b_{v_r j} = 0$. Similarly, $b_{ij} = 0$ for an edge $(i, j) \in \bar{E}$, $i, j \notin V_A$.

Therefore, it remains to evaluate b_{ij} only for the edges $(i, j) \in E_A$. Our goal is to show for each $z \in V_A$ that

$$b_{zi} = b_{zj} \quad \text{for each pair} \quad i, j \in N_z(A). \tag{11}$$

For $u \in V_A$, define $\bar{E}(u) = \{(u, i) \mid i \in N_u(A)\}$. Given $u \in V_A$, we construct a graph G_u with vertices t_{ui} corresponding to edges in $\bar{E}(u)$ and with edges added during a process to be described below. An edge (t_{ui}, t_{uj}) appears in G_u upon establishing the fact that $b_{ui} = b_{uj}$. Throughout this process, all edges will belong to only one connected component of G_u. We denote it by G^*_u. Upon termination of the process, G_u will appear to be connected. This fact will imply (11) for $z = u$.

To show (11) for all vertices in V_A, we use mathematical induction. We can assume w.l.o.g. that $V_A = \{1, \ldots, h\}$. First consider a vertex u for which $c_r \neq u, r = 1, \ldots, q$ (at least vertices $h, h-1, \ldots, h-q+2$ are of such type). The fact that G_u is connected will be established again by applying induction. We iteratively examine vertex sets K_r, $r = 1, \ldots, q$. We denote by $V_u^*(r)$, $u \in V_A$, $r \in \{1, \ldots, q\}$, the vertex set of G_u^* upon termination of the first r iterations. The statement to be proved inductively is the following: for $r \in \{1, \ldots, q\}$, the vertices t_{uc_i}, $i = 1, \ldots, r$, t_{uv_i}, $i = 2, \ldots, r$, t_{uw_i}, $i = 1, \ldots, r-1$, and, if $r < q$, also t_{uw_r} belong to $V_u^*(r)$.

In the first iteration, we consider K_1. Clearly, $c_1 \in \{v_i \mid i = 2, \ldots, q\}$. Comparing $E(s) = E(K_1) \cup E(\bar{K}_1'')$ and $E(s) \cup \{(u, w_1)\} \setminus \{(u, c_1)\}$ we find that $b_{uc_1} = b_{uw_1}$. The edge (t_{uc_1}, t_{uw_1}) defines initial G_u^*.

Suppose that $1 < r \leqslant q$. By the induction hypothesis, at the beginning of the rth iteration, G_u consists of the component induced by $V_u^*(r-1)$ and a cloud of vertices. Comparing $E(s) = E(K_r) \cup E(\bar{K}_r')$ and $E(s) \cup \{(u, v_r)\} \setminus \{(u, c_r)\}$ we conclude that $b_{uc_r} = b_{uv_r}$. If $r < q$, then similarly $b_{uc_r} = b_{uw_r}$. We add to G_u the edge (t_{uc_r}, t_{uv_r}) and, if $r < q$, also the edge (t_{uc_r}, t_{uw_r}). We can see that (at least) one of the vertices in $V_u^r := \{t_{uc_r}, t_{uv_r}, t_{uw_r}\}$ (V_u^r is without t_{uw_r} if $r = q$) already belongs to $V_u^*(r-1)$. Indeed, if $c_r \in \{w_i \mid i = 1, \ldots, r-1\}$, then $t_{uc_r} \in V_u^*(r-1)$. Suppose $c_r \in \{v_i \mid i = r+1, \ldots, q\}$. Then $c_{r-1} = c_r$ if $v_r > c_r$ and $c_{r-1} = v_r$ if $v_r < c_r$. In the first case, $t_{uc_r} = t_{uc_{r-1}} \in V_u^*(r-1)$ and, in the second case, $t_{uv_r} \in V_u^*(r-1)$. The fact that $V_u^r \cap V_u^*(r-1)$ is nonempty implies that $V_u^r \subseteq V_u^*(r)$.

At the end of the described iterative process, G_u coincides with G_u^* and therefore is connected. Thus (11) for $z = u$ is proved.

Suppose that (11) holds for $z = u+1, u+2, \ldots, h$, and now the vertex u is such that $c_r = u$ for at least one $r \in \{1, \ldots, q\}$. In this case, the above described process shall be modified. If, for $r \in \{1, \ldots, q\}$, $c_r \neq u$, then, at the rth iteration, the same arguments as before are used. So assume $r \in \{1, \ldots, q\}$ is such that $c_r = u$. For v_r, the following two cases are possible.

Case 1. $v_r > u, r > 1$. Since $c_r = u$ it follows that $w_i > u$, $i = 1, \ldots, r-1$, and $v_j > u$, $j = r+1, \ldots, q$. Suppose $r > 2$. According to the induction hypothesis, (11) holds for $w_1 > u$ and $v_r > u$. Therefore, $b_{w_1 u} = b_{w_1 v_r}$ and $b_{v_r u} = b_{v_r w_1}$. Consequently, $b_{w_1 u} = b_{v_r u}$ and $t_{uv_r} \in V_u^*(r)$. If $r = 2$, then the same conclusion is drawn from the equations $b_{v_3 v_2} = b_{v_3 w_1}$, $b_{w_1 u} = b_{w_1 v_3}$ and $b_{v_2 u} = b_{v_2 v_3}$ (v_3 exists when $h \geqslant 7$).

Case 2. $v_r < u, r > 1$. Then $c_{r-1} = v_r$ and, exactly as in the case of u for which $c_i \neq u$, $i = 1, \ldots, q$, we have $b_{uv_r} = b_{uv_{r-1}} = b_{uw_{r-1}}$. Hence $t_{uv_r} \in V_u^*(r-1) \subseteq V_u^*(r)$.

Notice that Case 2 can occur at most once. Indeed, $c_r = u$ implies that $v_j > u$, $j = r+1, \ldots, q$.

Similarly, for w_r, we consider the following two cases.

Case 1. $w_r > u, r < q$. Suppose $r > 1$. Then, applying (11) to $w_1 > u$ and w_r, we have $b_{w_1 w_r} = b_{w_1 u}$ and $b_{w_r w_1} = b_{w_r u}$. Consequently, $b_{w_1 u} = b_{w_r u}$ and $t_{uw_r} \in V_u^*(r)$. If $r = 1$, then $V_u^*(1) = \{t_{uw_1}\}$.

Case 2. $w_r < u, r < q$. Then $c_{r+1} = w_r$. By comparing $E(s) = E(K_{r+1}) \cup E(\bar{K}_{r+1}')$ and $E(s) \cup \{(u, v_{r+1})\} \setminus \{(u, w_r)\}$ we find that $b_{uv_{r+1}} = b_{uw_r}$. Suppose $r > 1$ (otherwise $V_u^*(1) = \{t_{uw_1}\}$). Then $w_1 > u$, $v_{r+1} > u$ and, analogously as in the above-considered cases, from (11) it follows that $b_{uv_{r+1}} = b_{uw_1}$. Thus $t_{uw_r} \in V_u^*(r)$.

Again, Case 2 can be encountered at most once. Indeed, for $j > r$, the requirement $w_i > u$, $i = 1, \ldots, j-1$, for u to be the centre vertex of the star $S(K_j)$ cannot be met.

Thus, we have proved that (11) holds for each $z \in V_A$. Since A is connected it follows that $b_{ij} = \lambda$ for all $(i, j) \in E_A$ and some $\lambda \in \mathbb{R}$. This means that $b^T x \leqslant b_0$ is a multiple of (10). Therefore, the inequality (10) is facet-defining for $P(G)$. □

3. Concluding remarks

In this paper, we presented several classes of facet-defining inequalities for the graph coloring polytope. Most of the proofs were based on the fact that this polytope coincides with the stable set polytope of the graph derived from the complement of a given graph. This relationship can be used to discover new classes of valid or even facet-defining inequalities. Another possible direction of further work is to devise efficient separation algorithms for such inequalities. We are hopeful that the results obtained will be of value in the development of new graph coloring algorithms.

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Received January 2008.