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SIMULIATION OF THE OPEN MESSAGE SWITCHING SYSTEM

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Abstract. The modern queueing theory is one of the powerful tools for a quantitative and qualitative analysis of communication systems, computer networks, transportation systems, and many other technical systems. The paper is designated to the analysis of queueing systems arising in the network theory and communications theory (called open queueing network). We present a theorem on the law of the iterated logarithm (LIL) for the virtual waiting time of a customer in an open queueing network and its application to the mathematical model of the open message switching system.

Keywords: Models of information systems, open queueing network, openmessage switching system, virtual waiting.

1. Statement of the Problem

At first the author focused on the research in queueing theory on the LIL in queueing systems and present a theorem for the virtual waiting time of a customer in an open queueing network. Note that the research of the LIL in more general systems than the classical queueing system $GI/G/1$ or multiphase queueing systems and open queueing network has just started (see (Whitt 2002)). In (Minkevičius 1995; 1997), the LIL has proved in heavy traffic for the queue length of customers, waiting time of a customer, and a virtual waiting time of a customer in a multiphase queueing system. In (Sakalauskas and Minkevičius 2000), the authors also gave the proof of the theorem on the LIL under the conditions of heavy traffic for a virtual waiting time of a customer in the open Jackson network.

In this paper, we investigated an open queueing network model in heavy traffic. We preent the LIL for the virtual waiting time of a customer in an open queueing network. The main tool for the analysis of these queueing systems in heavy traffic is a functional LIL for the renewal process (the proof can be found in (Strassen 1964) and (Iglehart 1971)).

The service discipline is "first come, first served" (FCFS). We consider open queueing networks with the FCFS service discipline at each station and general distributions of interarrival and service times. We study the queueing network with k single server stations, each of which has an associated infinite capacity waiting room. Every station has an arrival stream from outside the network, and the arrival streams are assumed to be mutually independent renewal processes. Customers are served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. Service times and routing decisions form mutually independent sequences of independent identically distributed random variables.

The basic components of the queueing network are arrival processes, service processes, and routing processes. In particular, there are mutually independent sequences of independent identically distributed random variables $\left\{z_n^{(j)}, n \geq 1\right\}$, $\left\{S_n^{(j)}, n \geq 1\right\}$ and $\left\{\Phi_n^{(j)}, n \geq 1\right\}$ for $j = 1, 2, \ldots, k$; defined on the probability space. Random variables $z_n^{(j)}$ and $S_n^{(j)}$ are strictly positive, and $\Phi_n^{(j)}$ have support in $\{0, 1, 2, \ldots, k\}$. We define $\mu_j = \left(M\left[S_n^{(j)}\right]\right)^{-1} > 0$, $\sigma_j = D\left(S_n^{(j)}\right) > 0$, $\lambda_j = \left(M\left[z_n^{(j)}\right]\right)^{-1} > 0, and\ a_j = D\left(z_n^{(j)}\right) > 0$, $j = 1, 2, ..., k$; with all of these terms assumed to be finite. Denote $p_{ij} = P\left(\Phi_n^{(i)} = j\right) > 0$, $j = 1, 2, \ldots, k$. In the context of the queueing network, the random variables $z_n^{(j)}$ function as interarrival times (from outside the network) at the station j, while $S_n^{(j)}$ is the nth service time at the station j, and $\Phi_n^{(j)}$ is a routing indicator for the nth customer served at the station j. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the nth customer served at the station i is routed to the station j. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network. The matrix P is called a routing matrix.

Observe that this system is quite general, encompassing the tandem system, acyclic networks of $GI/G/1$ queues, networks of $GI/G/1$ queues with feedback and an open queueing network.

First, let us define $\hat{V}_j(t)$ as the virtual waiting time of a customer at the j-th station of the queueing network at time t (the time one must wait until a customer arrives at the j-th station of the queueing network to be served at time t), $\hat{\beta}_j = \frac{\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{i,j}}{\mu_j}$

$-1 > 0, \quad \hat{\sigma}_j^2 = \sum_{i=1}^{k} (p_{i,j})^2 \cdot \mu_i \left(\sigma_j + \left(\frac{\mu_i}{\mu_j} \right)^2 \cdot \sigma_i \right)$

$+\lambda_j \cdot \left(\sigma_j + \left(\frac{\lambda_j}{\mu_j} \right)^2 \cdot a_j \right) > 0, \; j = 1, 2, \ldots, k.$

Suppose that the virtual waiting time of a customer in each station of the open queueing network is unlimited. All random variables are defined on one common probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

We assume the following condition is fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} > \mu_j, \; j = 1, 2, \ldots, k. \quad (1)$$

Note that this condition quarantees that, with probability one, there exists a virtual waiting time of a customer and this virtual waiting time of a customer is constantly growing.

2. The Main Result

One of the results of the paper is the following theorem on the LIL for the virtual waiting time of a customer in an open queueing network.

Theorem 1. *If conditions (1) are fulfilled, then*

$$P\left(\overline{\lim_{t \to \infty}} \frac{\hat{V}_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = 1 \right)$$

$$= P\left(\underline{\lim_{t \to \infty}} \frac{\hat{V}_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = -1 \right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(t) = \sqrt{2t \ln \ln t}.$

Proof. This theorem is proved under the conditions $\lambda_j > \mu_j, \quad j = 1, 2, \ldots, k$ (see, for example, (Sakalauskas, Minkevičius 2000)). Applying the methods of Minkevičius, Kulvietis (2007), we can prove that this theorem is true under more general conditions (1).

The proof of the theorem is completed.

3. On the Model of the Open Message Switching Facility

In this section, we present an application of the proved theorem - a mathematical model of an open message switching system. As noted in the introduction, open network queueing systems are of special interest both in theory and in practical applications. Such systems consist of several service nodes, and each arriving customer is served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. A typical example is provided by queueing systems with identical service. Such systems are very important in applications, especially to open message switching systems. In fact, in many comunication systems the transmission times of customers do not vary in the delivery process.

So, we investigate a message switching system which consists of k service nodes and in which $S_n^{(j)} = S_n, \quad j = 1, \; 2, \ldots, k$ (the service process is identical at the nodes of the system).

Next, denote $\mu = (M\,[S_n])^{-1} > 0, \quad \beta_j = \frac{\lambda_j + \mu \cdot (\sum_{i=1}^{k} p_{i,j})}{\mu} - 1 > 0, \; \tilde{\sigma}_j^2 = \sum_{i=1}^{k} (p_{i,j})^2 \cdot (2 \cdot \mu \cdot D(S_n)) + \lambda_j \cdot \left(\mu + (\frac{\lambda_j}{\mu})^2 \cdot D(z_n^{(j)}) \right) > 0, \; j = 1, 2, \ldots, k.$

We assume that the following conditions are fulfilled:

$$\beta_j > 0, j = 1, 2, \ldots, k. \quad (2)$$

Similarly as in the proof of Theorem 2.1, we present the following theorem and corollary on the LIL for the virtual waiting time of messages in open message switching systems.

Theorem 2. *If conditions (2) are fulfilled, then*

$$P\left(\overline{\lim_{t \to \infty}} \frac{V_j(t) - \beta_j \cdot t}{\tilde{\sigma}_j \cdot a(t)} = 1 \right)$$

$$= P\left(\underline{\lim_{t \to \infty}} \frac{V_j(t) - \beta_j \cdot t}{\tilde{\sigma}_j \cdot a(t)} = -1 \right) = 1,$$

$j = 1, 2, \ldots, k$ and $a(t) = \sqrt{2t \ln \ln t}.$

Corollary 1. *If conditions (2) are fulfilled, then for fixed* $\varepsilon > 0$ *there exists* $t(\varepsilon)$ *such that for every* $t \geq t(\varepsilon)$,

$$(1 - \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t) + \beta_j \cdot t \leq V_j(t) \leq$$

$$(1 + \varepsilon) \cdot \tilde{\sigma}_j \cdot a(t) + \beta_j \cdot t, \;\; j = 1, 2, \ldots, k,$$

with probability one.

4. On the Model of the Nack Type Switching Facility

Consider a switching facility that transmits messages to a required destination. A NACK (Negative ACKnowledgement) is sent by the destination when a packet has not been properly transmitted. If so, the packet in error is retransmitted as soon as the NACK is received.

Now we present the investigation of a separate case of Theorem 3.1 (the case of the NACK type message swiching system). We now assume that the switching facility is composed of k nodes in series, each modeled as a $G/GI/1$ queue with the common service rate μ. In other words, we now have an open Jackson network with k $G/GI/1$ queues where $\lambda_j = 0$ for $i = 2, 3, \ldots, k$ (no external arrivals at nodes $2, 3, \ldots, k$), $\mu_i = \mu$ for $i = 2, 3, \ldots, k$, $p_{i,i+1} = 1$ for $i = 1, 2, \ldots, k-1$, $p_{k,0} = p$ and $p_{k,1} = 1 - p$.

So, we investigate a NACK type switching system which consists of k service nodes and in which $S_n^{(j)} = S_n$, $j = 1, 2, \ldots, k$ (the service process is identical in the phases of the system).

Next, denote $\bar{V}_j(t)$ as the virtual waiting time of messages in the j-th phase of the NACK type message switching system at the time moment t; $j = 1, 2, \ldots, k$ and $t > 0$.

Let us define $\bar{\beta}_1 = \dfrac{\lambda_1 + \mu \cdot p_{1,2}}{\mu} - 1 = \dfrac{\lambda_1 + \mu}{\mu} - 1 = \dfrac{\lambda_1}{\mu}$, $\bar{\sigma}_1^2 = (p_{1,2})^2 \cdot (2 \cdot \mu D(S_n)) + \lambda_1 \cdot (\mu + (\dfrac{\lambda_1}{\mu})^2 \cdot D(z_n^{(1)})) = (2 \cdot \mu \cdot D(S_n)) + \lambda_1 \cdot (\mu + (\dfrac{\lambda_1}{\mu}) \cdot D(z_n^{(1)})) > 0$, $\bar{\beta}_j = \dfrac{\mu \cdot (p_{j,j+1})}{\mu} - 1 = p_{j,j+1} - 1 = 0$, $\bar{\sigma}_j^2 = (p_{j,j+1})^2 \cdot (2 \cdot \mu \cdot D(S_n)) = 2 \cdot \mu \cdot D(S_n) > 0$, $j = 2, 3, \ldots, k-1$, $\bar{\beta}_k = \dfrac{\lambda_1 + \mu \cdot p_{k,1}}{\mu} - 1 = \dfrac{\lambda_1 + \mu \cdot (1-p)}{\mu} - 1 = \dfrac{\lambda_1}{\mu} - p$, $\bar{\sigma}_k^2 = (p_{k,1})^2 \cdot (2 \cdot \mu D(S_n)) = (1-p)^2 \cdot (2 \cdot \mu D(S_n)) > 0$.

We assume that the following conditions are fulfilled:

$$\bar{\beta}_1 > 0,\ \bar{\beta}_k > 0,\ \bar{\beta}_j = 0,\ j = 2, 3, \ldots, k-1. \quad (3)$$

Applying Theorem 3.1, we present a theorem and corollary about the virtual waiting time of messages in the NACK type message switching system.

Theorem 3. *If conditions (3) are fulfilled, then*

$$P\Big(\overline{\lim_{n\to\infty}} \frac{\bar{V}_j(t) - \bar{\beta}_j \cdot t}{\bar{\sigma}_j \cdot a(n)} = 1 \Big) =$$

$$P\Big(\underline{\lim}_{n\to\infty} \frac{\bar{V}_j(t) - \bar{\beta}_j \cdot t}{\bar{\sigma}_j \cdot a(n)} = -1 \Big) = 1,$$

$j = 1, 2, \ldots, k$, *and* $a(n) = \sqrt{2n \ln \ln n}$.

Corollary 2. *If conditions (3) are fulfilled, then for fixed* $\varepsilon > 0$ *there exists* $t(\varepsilon)$ *such that for every* $t \geq t(\varepsilon)$

$$(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t \leq \bar{V}_j(t) \leq (1+\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t,$$

$j = 1, 2, \ldots, k$, *with probability one.*

5. Computing Example

We see that Corollary 4.1 implies that for fixed $\varepsilon > 0$ there exists $t(\varepsilon)$ such that for every $t \geq t(\varepsilon)$,

$$(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t \leq \bar{V}_j(t) \leq (1+\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t,$$

where $a(t) = \sqrt{2t \ln \ln t}$, $\bar{\beta}_1 = \dfrac{\lambda_1}{\mu}$, $\bar{\sigma}_1^2 = (2 \cdot \mu \cdot D(S_n)) + \lambda_1 \cdot (\mu + (\dfrac{\lambda_1}{\mu}) \cdot D(z_n^{(1)})) > 0$, $\bar{\beta}_j = 0$, $\bar{\sigma}_j^2 = (2 \cdot \mu \cdot D(S_n)) > 0$, $j = 2, 3, \ldots, k-1$; $\bar{\beta}_k = \dfrac{\lambda_1}{\mu} - p$, $\bar{\sigma}_k^2 = (1-p)^2 \cdot (2 \cdot \mu \cdot D(S_n)) > 0$, $\varepsilon > 0$, $t > 0$.

From this we can obtain

$$(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t \leq MV_j(t) \leq (1+\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) + \bar{\beta}_j \cdot t$$

$$|M(V_j(t) - \bar{\beta}_j \cdot t) - \{(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t)\}| \leq 2 \cdot \varepsilon \cdot \bar{\sigma}_j \cdot a(t)$$

$$\left| M\left(\frac{\bar{V}_j(t) - \bar{\beta}_j \cdot t)}{\bar{\sigma}_j \cdot a(t)} \right) - (1-\varepsilon) \right| \leq 2 \cdot \varepsilon,\ j = 1, 2, \ldots, k. \quad (4)$$

Thus, it follows from (4) that

$$M\bar{V}_j(t) \sim \bar{\beta}_j \cdot t + (1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t),\ j = 1, 2, \ldots, k. \quad (5)$$

$M\bar{V}_j(t)$ is the average virtual waiting time of messages in the NACK type message switching system at the time moment t, $j = 1, 2, \ldots, k$ and $t > 0$.

We see from (5) that $M\bar{V}_j(t)$ consists of the linear function $\bar{\beta}_j \cdot t$ and a nonlinear slowly increasing function $(1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t)$, $j = 1, 2, \ldots, k$ and $t > 0$.

Now we present a technical example from the computer network practice. Assume that messages arrive at the computer V_1 at the rate λ_1 of 21 per hour during business hours. These messages are served at the rate μ of 20 per hour in the computer V_1.

After service in the computer V_1 messages arrive at the second computer V_2. Also note that the messages are served at the rate μ of 20 per hour in the computer V_2. So, messages are served in the computers $V_1, V_2, \ldots, V_k$, and after they are served in the computer V_k, with the probability $p = 0.9$ (probability that a message is received correctly), they leave the computer network and are sent to the computer V_1 with probability $1 - p = 0.1$.

So, $\bar{\beta}_1 = \dfrac{\lambda_1}{\mu} = \dfrac{21}{20} = 1.05$, $\bar{\sigma}_1^2 = (2 \cdot \mu \cdot D(S_n)) + \lambda_1 \cdot (\mu + (\dfrac{\lambda_1}{\mu})^2 \cdot D(z_n^{(1)})) = 422.01$, $\bar{\sigma}_1 = 20.5428$, $\bar{\beta}_j = 0$, $\bar{\sigma}_j^2 = 2 \cdot \mu \cdot D(S_n) = 2$, $\bar{\sigma}_j = 1.41$, $j = 2, 3, \ldots, k-1$, $\bar{\beta}_k = \dfrac{\lambda_1}{\mu} - p = 0.15$, $\bar{\sigma}_k^2 = (1-p)^2 \cdot (2 \cdot \mu D(S_n)) = 0.045$, $\bar{\sigma}_k = 0.2121$, $\varepsilon = 0.001$, $t \geq 100$.

Thus,

$$M\bar{V}_1(t) \sim \bar{\beta}_j \cdot t + (1-\varepsilon) \cdot \bar{\sigma}_j \cdot a(t) = (1.05) \cdot t + (20.5222) \cdot a(t). \tag{6}$$

From (6) we get

$$\frac{M\bar{V}_1(t)}{t} = 1.05 + (20.52) \cdot \sqrt{\frac{2 \ln \ln t}{t}}. \tag{7}$$

Similarly as in (7) we can obtain

$$\frac{M\bar{V}_j(t)}{t} = (1.41) \cdot \sqrt{\frac{2 \ln \ln t}{t}}, \quad j = 2, 3, \ldots, k-1 \tag{8}$$

and

$$\frac{M\bar{V}_k(t)}{t} = (0.15) + (0.21) \cdot \sqrt{\frac{2 \ln \ln t}{t}}. \tag{9}$$

Now we present figures for $\dfrac{M\bar{V}_j(t)}{t}$, $j = 1, 2, \ldots, k$, when $100 \leq t \leq 1000$, $\varepsilon = 0.001$ (see (7) - (9) and Table 1)

Table 1 Summary of computing results

$Time\ t$	$\dfrac{M\bar{V}_1(t)}{t}$	$\dfrac{M\bar{V}_j(t)}{t}$, $j = 2, \ldots, k-1$	$\dfrac{M\bar{V}_k(t)}{t}$
100	4.6366	0.24690	0.18702
200	3.7000	0.18242	0.17735
300	3.2610	0.15220	0.17282
400	2.9916	0.13366	0.17004
500	2.8043	0.12076	0.16811
600	2.6641	0.11111	0.16666
700	2.5539,	0.10353	0.16552
800	2.4643	0.09735	0.16460
900	2.3895	0.09221	0.16382
1000	2.3259	0.08783	0.16317

Corollary 3. *When $\bar{\beta}_j \geq 0, j = 1, 2, 3, \ldots, k$, the average virtual waiting time of messages is small at all nodes of the open message system.*

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