AVERAGED RELATIVE ERROR NORMS FOR VALIDATION OF EIGENMODE CALCULATIONS

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Abstract. The static problem of torsion is analysed for comparison of various expressions of error norms. On the basis of this analysis the expression for the relative error norm is chosen. The eigenproblem of plane stress is analysed. For validation of eigenmode calculations averaged relative error norms are introduced. The representation of relative error norms and of averaged relative error norms by the intensities of finite elements with intensity mapping for better utilisation of the intensity scale is proposed. The presented results serve as a means for validation of eigenmode calculations.

Keywords: computational validation, error norms, finite elements

1. Indroduction

The question of reliability of computer generated solutions is one of the concerns to specialists in computational engineering. Error estimations provide a quantitative measure for determining the quality of numerical simulations. They provide a basis for adapting characteristics of discrete models (meshes, approximation orders, etc) so as to improve the quality of results. The validation of computational models [1], [2] in finite element calculations is usually performed by using the error norms [3], [4] of finite elements. The static problem of torsion is analysed for comparison of various expressions of error norms. On the basis of this analysis the expression for the relative error norm is chosen.

When analysing the vibrations of structures a number of first eigenmodes is usually required. They are all calculated on the same finite element mesh. For validation of such calculations averaged relative error norms are introduced in this paper. The problem of plane stress [3], [5], [6] is analysed using the conventional finite element displacement formulation. The presented results are applicable to the problems described in [7], [8].

2. Analysis of expressions for the finite element error norms

For comparative analysis of various expressions of error norms the static problem of torsion is solved because of its simplicity. In this problem the displacements are assumed to be [5], [6]:

$$\begin{aligned} u &= -yz\theta, \\ v &= xz\theta, \\ \tilde{w} &= w(x, y)\theta, \end{aligned} \tag{1}$$

where u, v, $\tilde{w}$ are the components of the displacement vector in the directions of the x, y, z axes of the orthogonal Cartesian co-ordinate system, θ is the angle of twist per unit length. The strains are expressed as:

$$\begin{aligned} \gamma_{xz} &= \left(\frac{\partial w}{\partial x} - y\right)\theta, \\ \gamma_{yz} &= \left(\frac{\partial w}{\partial y} + x\right)\theta. \end{aligned} \tag{2}$$

So the strains per unit angle of twist $\{\varepsilon\}$ are expressed as:

$$\{\varepsilon\} = [B]\{\delta\} + \begin{Bmatrix} -y \\ x \end{Bmatrix}, \tag{3}$$

where $[B]$ is the matrix of derivatives of the shape functions (the first row with respect to x and the second row with respect to y), $\{\delta\}$ is the vector of nodal values of $w(x, y)$.

Thus the stiffness matrix takes the form:

$$[K]=\int [B]^T [D][B]dxdy, \tag{4}$$

where the matrix of elastic constants:

$$[D]=\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}, \tag{5}$$

where G is the shear modulus. The loading vector takes the form:

$$\{F\}=\int [B]^T [D]\begin{Bmatrix} y \\ -x \end{Bmatrix}dxdy. \tag{6}$$

The components of stresses in the domain of the analysed finite element can be calculated in the usual way [3], [5], [6]:

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}=[D]\{\varepsilon\}. \tag{7}$$

The displacements are continuous at inter-element boundaries, but the calculated stresses are discontinuous due to the operation of differentiation. The nodal values of the components of stresses are obtained by using the conjugate approximation [5].

The components of the stresses can be interpolated from their nodal values by using the shape functions of the finite element. Then the components of strains are obtained using those values of stresses and the matrix of elastic constants:

$$\{\varepsilon^*\}=[D]^{-1}\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}. \tag{8}$$

The absolute error norm [3] for the i-th finite element then can be calculated as:

$$\psi_i^{abs}=\iint_{e_i}\left(\{\varepsilon^*\}-\{\varepsilon\}\right)^T [D]\left(\{\varepsilon^*\}-\{\varepsilon\}\right)dxdy. \tag{9}$$

The estimate of the quantity proportional to the potential energy of the finite element for the i-th finite element can be calculated on the basis of the values obtained by conjugate approximation as:

$$\Pi_i^*=\iint_{e_i}\{\varepsilon^*\}^T [D]\{\varepsilon^*\}dxdy. \tag{10}$$

The estimate of the quantity proportional to the potential energy of the finite element for the i-th finite element can also be calculated as:

$$\Pi_i=\iint_{e_i}\{\varepsilon\}^T [D]\{\varepsilon\}dxdy. \tag{11}$$

The relative error norm [3] for the i-th finite element then can be calculated as:

$$\psi_i^*=\frac{\psi_i^{abs}}{\Pi_i^*}, \tag{12}$$

or as:

$$\psi_i=\frac{\psi_i^{abs}}{\Pi_i}. \tag{13}$$

Typical sequence of calculations is presented in Figure 1.

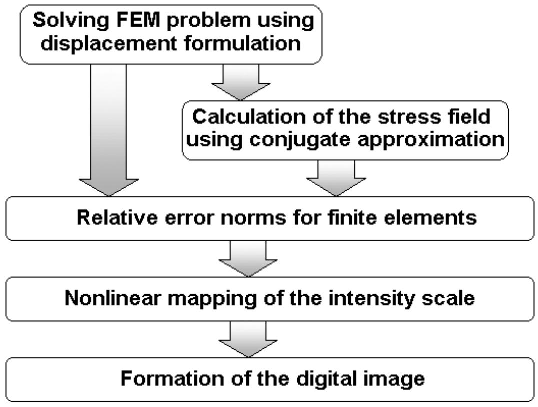


Figure 1. The sequence of calculations

3. Numerical simulation

A rectangular cross section is analysed. The absolute error norm (9), the estimates of the quantity proportional to the potential energy of the finite element (10), (11) and the relative error norms (12), (13) are represented by the intensity of the finite elements. In order to more fully utilise the intensity values the following mapping is proposed:

$$(\psi_i^{abs})_{mapped}=\left(\frac{\psi_i^{abs}-\min\limits_i \psi_i^{abs}}{\max\limits_i \psi_i^{abs}-\min\limits_i \psi_i^{abs}}\right)^m, \tag{14}$$

$$(\Pi_i^*)_{mapped}=\left(\frac{\Pi_i^*-\min\limits_i \Pi_i^*}{\max\limits_i \Pi_i^*-\min\limits_i \Pi_i^*}\right)^m, \tag{15}$$

$$(\Pi_i)_{mapped}=\left(\frac{\Pi_i-\min\limits_i \Pi_i}{\max\limits_i \Pi_i-\min\limits_i \Pi_i}\right)^m, \tag{16}$$

$$(\psi_i^*)_{mapped}=\left(\frac{\psi_i^*-\min\limits_i \psi_i^*}{\max\limits_i \psi_i^*-\min\limits_i \psi_i^*}\right)^m, \tag{17}$$

$$(\psi_i)_{mapped}=\left(\frac{\psi_i-\min\limits_i \psi_i}{\max\limits_i \psi_i-\min\limits_i \psi_i}\right)^m, \tag{18}$$

where m is the mapping parameter. The value of m = 0.125 was used in the further representations for the non-linear transformation of the intensity scale (Figure 2).

The calculated results are shown in Figure 3, Figure 4 and Figure 5.

In Table 1 the minimum and maximum values of the represented quantities are given.

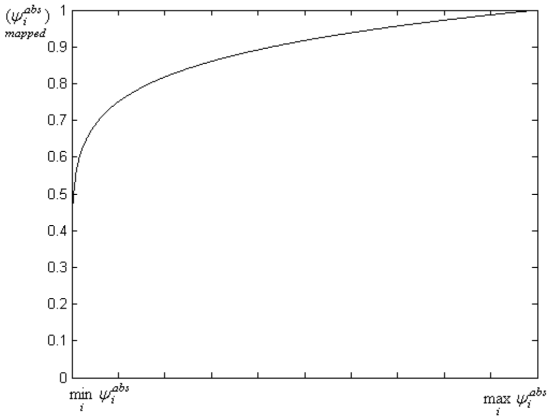


Figure 2. Non-linear transformation of the intensity scale

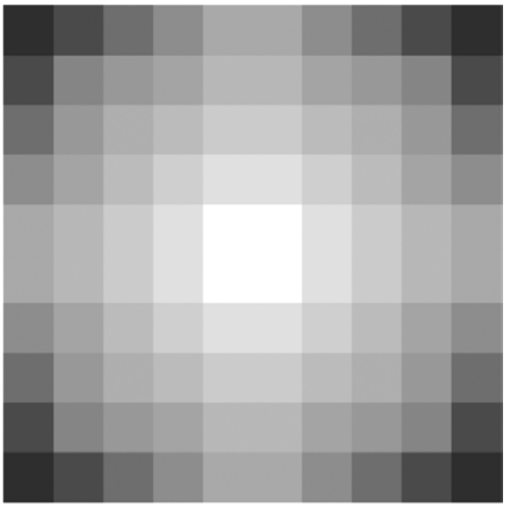


Figure 3. Absolute error norms

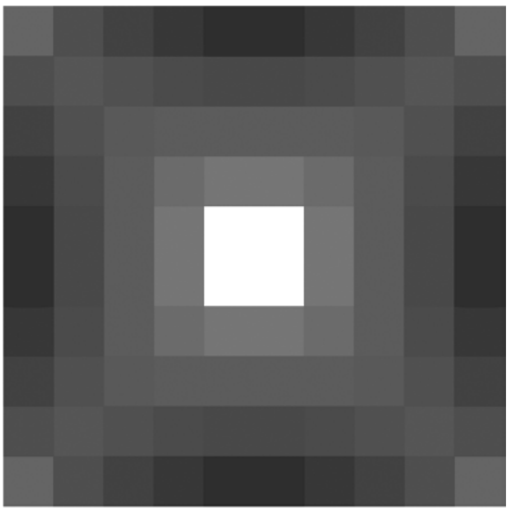


Figure 4. Estimate of the potential energy Π_i^* ($\Pi_i \approx \Pi_i^*$)

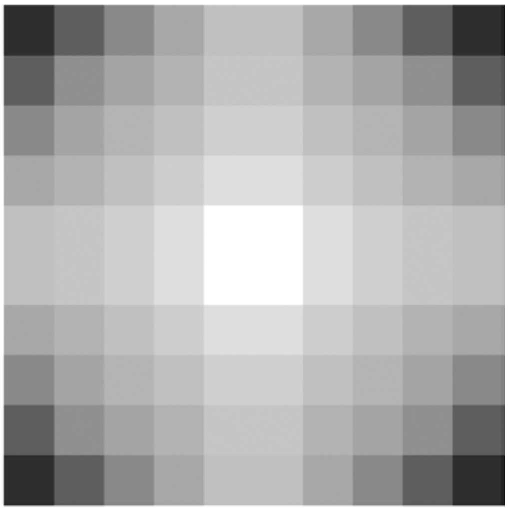


Figure 5. Relative error norms ψ_i^* ($\psi_i \approx \psi_i^*$)

The presented results show that the values of both relative error norms are similar and the choice of the relative error norm used further is determined by the effectiveness of numerical calculations.

Table 1. Minimum and maximum values of the error norms and estimates of the quantity proportional to the potential energy of the finite element

Quantity	$\min_i$	$\max_i$
ψ_i^{abs}	6.78453e-10	0.000363896
Π_i^*	0.83117	42.097
Π_i	0.831186	42.0927
ψ_i^*	8.16262e-10	5.39726e-05
ψ_i	8.16247e-10	5.36212e-05

4. Averaged relative error norms for validation of eigenmode calculations

The problem of plane stress is analysed further. The components of stresses in the domain of the analysed finite element can be calculated in the usual way [3], [5], [6]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D][B]\{\delta_0\}, \qquad (19)$$

where $\{\delta_0\}$ is the vector of nodal displacements of the eigenmode; $[B]$ is the matrix relating the strains with the displacements; $[D]$ is the matrix relating the stresses with the strains; σ_x, σ_y, τ_{xy} are the components of the stresses in the problem of plane stress. It can be noted that the displacements are continuous at inter-element boundaries, but the calculated stresses are discontinuous due to the operation of differentiation.

The nodal values of the components of stresses for each eigenmode are obtained by using the conjugate approximation [5].

The components of the stresses can be interpolated from their nodal values by using the shape functions of the finite element. Then the components of strains ε_x, ε_y, γ_{xy} are obtained using those values of stresses and the matrix of elastic constants:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [D]^{-1} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}. \qquad (20)$$

The relative error norm [3] for the i-th finite element then can be calculated as:

$$\psi_i = \frac{\iint_{e_i} (\{\varepsilon\} - [B]\{\delta_0\})^T [D] (\{\varepsilon\} - [B]\{\delta_0\}) dxdy}{\iint_{e_i} \{\varepsilon\}^T [D] \{\varepsilon\} dxdy}. \qquad (21)$$

The described calculations are performed for each of the required first eigenmodes and further the value of ψ_i for eigenmode j is denoted by ψ_i^j. The averaged relative error norms are defined as:

$$\overline{\psi}_i = \frac{\sum_{j=1}^{n} \psi_i^j}{n}, \qquad (22)$$

where n is the number of the first eigenmodes which are taken into account in the analysis.

5. Numerical results

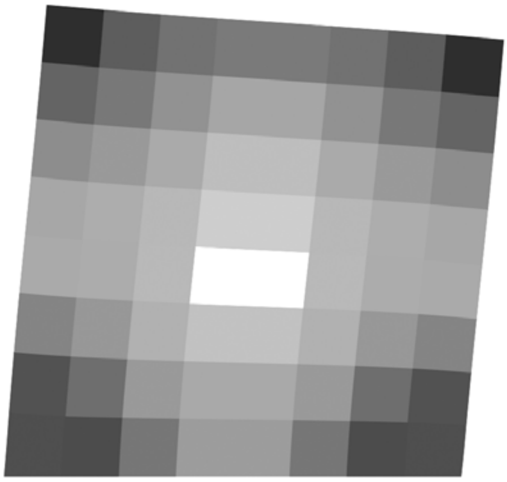
A rectangular plate with fixed edge in the state of plane stress is analysed. The lower edge of the plate is fastened (both components of displacements are assumed equal to zero). It is considered that the plate is experiencing resonant vibrations on an eigenmode which is not multiple: the loading is assumed to be harmonic with the frequency of the eigenmode and not orthogonal to it. The motion according to a single eigenmode is analysed first. The relative error norms are represented by the intensity of the finite elements. In order to more fully utilise the intensity values the following mapping is proposed:

$$(\psi_i^j)_{mapped} = \left(\frac{\psi_i^j - \min_i \psi_i^j}{\max_i \psi_i^j - \min_i \psi_i^j} \right)^m, \qquad (23)$$

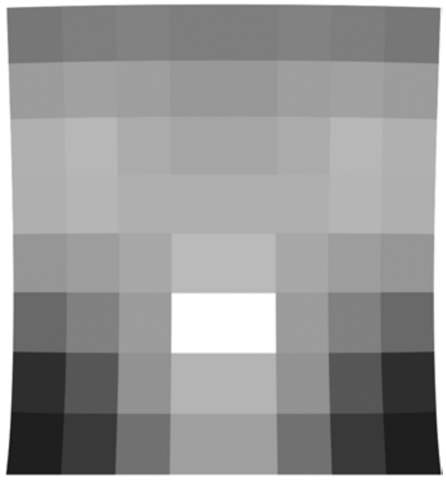
where m is the mapping parameter. Also:

$$(\overline{\psi}_i)_{mapped} = \left(\frac{\overline{\psi}_i - \min_i \overline{\psi}_i}{\max_i \overline{\psi}_i - \min_i \overline{\psi}_i} \right)^m. \qquad (24)$$

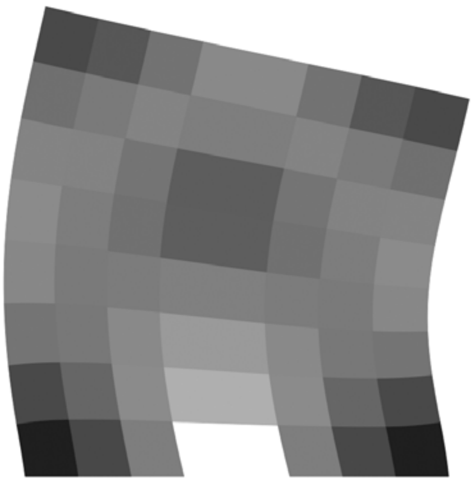
The value of $m = 0.132$ was used in the further representations. The relative error norms for the first six eigenmodes of the structure simultaneously showing the shape of the eigenmodes are shown in Figure 6. The averaged relative error norms for the first six eigenmodes are shown in Figure 7.



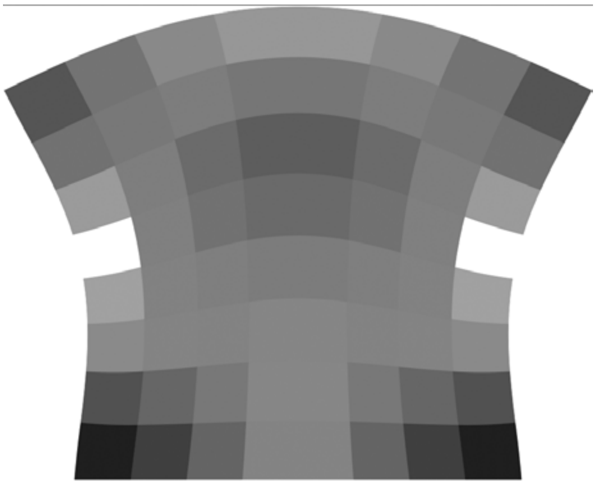
a)



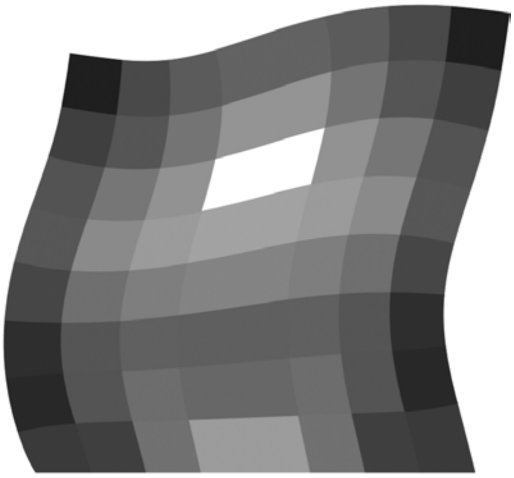
b)



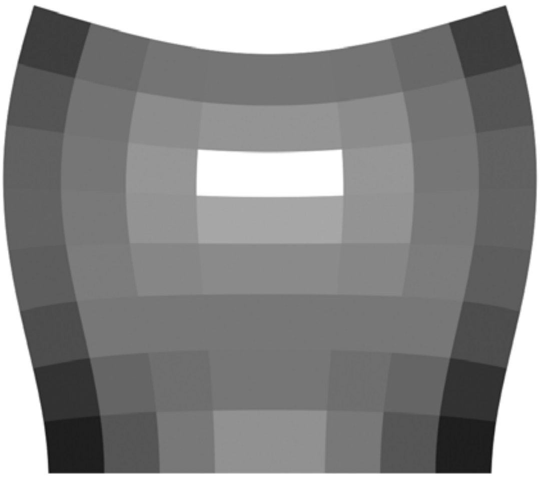
c)



d)



e)



f)

Figure 6. Relative error norms for the a) first, b) second, c) third, d) fourth, e) fifth, f) sixth eigenmodes

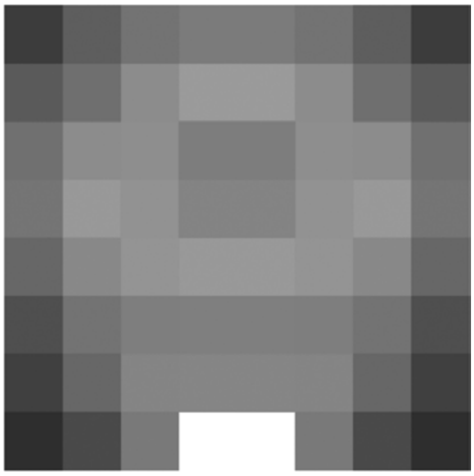


Figure 7. Averaged relative error norms for the first six eigenmodes

Table 2. Minimum and maximum values of the relative error norms and averaged relative error norms

Index of eigenmode j	$\min_i \psi_i^j$	$\max_i \psi_i^j$
1	1.29745e-07	0.00079013
2	1.27686e-07	0.00022923
3	9.65636e-07	0.00079498
4	4.31113e-06	0.000825917
5	3.62946e-06	0.00138171
6	1.86125e-06	0.000424316
	$\min_i \overline{\psi}_i$	$\max_i \overline{\psi}_i$
	5.50895e-06	0.000539685

The results presented in the figures together with the values presented in the table serve for validation of the eigenmode calculations.

6. Conclusions

The static finite element problem of torsion is analysed for comparison of various expressions of error norms. On the basis of this analysis the expression for the relative error norm is chosen on the best estimate for numerical effectiveness of calculations.

The presented computational validation techniques are applied for dynamic finite element problems. The problem of plane stress is analysed. Averaged relative error norms are introduced for validation of eigenmode calculations. Nonlinear mapping of relative error norms and averaged relative error norms of finite elements is proposed for better utilisation of the intensity scale.

The presented results serve as a means for computational validation of eigenmode calculations.

References

[1] **J. Garcia.** The need for computational model validation. *Experimental Techniques, March/April* 2001, 31-33.

[2] **J.T. Oden., T. Belytschko, I. Babuska, T.J.R. Hughes.** Research directions in computational mechanics. *Comput. Methods Appl. Mech. Engrg.*, 192, 2003, 913-922.

[3] **R. Barauskas.** Basis of the finite element method. *Technologija: Kaunas*, 1998.

[4] **O.C. Zienkiewicz, K. Morgan.** Finite elements and approximation. *Mir: Moscow*, 1986.

[5] **L.J. Segerlind.** Applied Finite Element Analysis. *Mir: Moscow*, 1979.

[6] **K.J. Bathe.** Finite element procedures in engineering analysis. *Prentice-Hall: New Jersey*, 1982.

[7] **K. Ramesh, P.M. Pathak.** Role of photoelasticity in evolving discretization schemes for FE analysis. *Experimental Techniques, July/August* 1999, 36-38.

[8] **P. Avitabile, J.M. Hunnell, A. Frank, J. DeVlaminck.** Study of different analytical model configurations to account for manufacturing variability using model correlation tools. *Experimental Techniques, January/February* 2001, 46-51.