ENTROPY AND THE COMPLEXITY FOR ZN ACTIONS

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Abstract. The complexity of a finite object was introduced by A. Kolmogorov and V. Tihomirov in [1] and it was conjectured that for Z actions the complexity coincides with topological entropy, [1] [2], [3]. In the present paper we introduce complexity for $\boldsymbol{Z^n}$ actions and prove the Kolmogorov assertion for continuous actions of Z

Key words: Dynamical system, Configuration spaces, Complexity, Topological entropy.

Let us introduce definitions and notations we need.

Let $A = \{a_1, ..., a_k\}$ be a finite set of symbols, (alphabet);

$$\Omega = A^{Z^n} = w = \{(w_g) : w_g \in A, g \in Z\}$$

be the space of configurations with Tychonoff topology, σ be the shift in this configuration space:

$$(\sigma^g w)_h = w_{g^{-1}h,} g, h \in Z^n .$$

Definition 1. A dynamical system (X, T) is a symbolic system on Z^n, if X is the σ-invariant closed subset of Ω and T is the restriction of σ to X.

Now we define the complexity of the configuration spaces of the symbolic dynamical system (X, T).

Definition 2. For an arbitrary finite subset F of Z^n we denote by A^F the set of stamps (configuration) on F. Every point

$$w^F = (w_g, g \in F) \in A^F$$

on this set A^F is called a configuration stamp.

Let P be some program which acts from the set of all finite words in the $\{0, 1\}$-alphabet into the space of stamps. By $l(p)$ we denote the number of elements in the finite word p in the $\{0,1\}$-alphabet.

Now we define complexity $C_P(w^F)$ of the stamp w^F relatively to the program P:

$$C_P(w^F) = \begin{cases} \inf\{l(p) : P(p) = w^F\} & \text{if}\{p : P(p) = w^F\} \neq 0 \\ \infty & \text{if}\{p : P(p) = w^F\} = 0 \end{cases}$$

Now we define the complexity $C_P(w)$ for the configuration $w \in X$ relatively to the program P:

$$C_P(w^F) = \lim_{k\to\infty} \sup \frac{1}{|I_k|} C_P(w|_{I_k}),$$

where $I_k = \{(i_1, i_2, ..., i_n) \in Z^n : -k \le i_j \le k,\ j = 1, 2, ..., n\}$, $|I_k| = (2k + 1)^n$.

Now let $C_p(X)$ define complexity of the configuration space X relatively to the program P as:

$$C_P(X) = \lim_{k\to\infty} \sup \frac{1}{|I_k|} \sup_{w\in X} C_P(w|_{I_k}).$$

Let P be such a program that for an arbitrary program P' we have a constant $C(P, P')$ such that for every stamp w^F the inequality

$$C_P(w^F) \le C_{P'}(w^F) + C(P, P')$$

holds.

We call this program P the asymptotically optimal program.

The existence of such a program P was proved in [1].

Proposition 1: For every symbolic system (X, T) and arbitrary optimal programs P_1 and P_2,

$$C_{P_1}(X) = C_{P_2}(X).$$

Proof: Let us prove the inequality

$$C_{P_1}(X) \le C_{P_2}(X)$$

From the definition of an asymptoticaly optimal program we have for an arbitrary stamp w_F

$$C_{P_1}(X) \le C_{P_2}(X)(w_F) + C(P_1, P_2)$$

where $C(P_1, P_2)$ is a constant.

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Thus

$$\sup_{w\in X} C_{P_1}(w|_{I_k}) \le \sup_{w\in X} C_{P_2}(w|_{I_k}) + C(P_1, P_2)$$

and then

$$\frac{1}{|I_k|}\sup_{w\in X} C_{P_1}(w|_{I_k}) \le \frac{1}{|I_k|}\sup_{w\in X} C_{P_2}(w|_{I_k}) + $$
$$+\frac{1}{|I_k|}C(P_1, P_2)\,.$$

So

$$\lim_{k\to\infty}\frac{1}{|I_k|}\sup_{w\in X} C_{P_1}(w|_{I_k}) \le$$
$$\le \lim_{k\to\infty}\frac{1}{|I_k|}\sup_{w\in X} C_{P_2}(w|_{I_k}) +$$
$$+\lim_{k\to\infty}\frac{1}{|I_k|}C(P_1, P_2)\,.$$

But for every constant C we have:

$$\limsup_{k\to\infty}\frac{1}{(2k+1)^n}C = 0\,.$$

So $C_{P_1}(X) \le C_{P_2}(X)$.

Now we will prove the main results of our paper.

Theorem 1: Let (X, T) be a symbolic system on Z^n. Then

$$C_P(X) = h_t(T)\,,$$

where $h_t(T)$ is the topological entropy of the action T of the group Z^n on X.

Proof: Let the complexity $C_P(X)$ of the space X be finite and equal to a. So we have:

$$\limsup_{k\to\infty}\frac{1}{|I_k|}\sup_{w\in X} C_{P_1}(w|_{I_k}) = a\,.$$

Then let $\varepsilon > 0$ be an arbitrary number. There is some $n_0 \in IN$ such that $\forall k > n_0$

$$\frac{1}{|I_k|}\sup_{w\in X} C_P\,(w|_{I_k}) \le a+\varepsilon\,.$$

So we have

$$\sup_{w\in X} C_P\,(w|_{I_k}) \le (a+\varepsilon)\,|I_k|\,. \qquad (1)$$

The inequality shows us that the number of different restrictions of points of X on the I_k set is not bigger than $2^{(a+\varepsilon)|I_k|+1}$.

To prove this, we can write from the definition,

$$P:\bigcup_{n=1}^{\infty}\{0,1\}^n \to \bigcup_{\substack{F\subset Z\\ card\,F<\infty}} A^F$$

for any P program. Now we will find some set $\mathbf{U}$ such that

$$\mathbf{U}\subset\bigcup_{n=1}^{\infty}\{0,1\}^n \text{ and } P(\mathbf{U}) = V,$$

where

$$V = \{\overline{w} = (w_g, g\in I_k) : \exists \quad \widetilde{w}\notin X\,,\quad \widetilde{w}|_{I_k} = \overline{w}\} =$$
$$= A^{I_k}\cap X|_{I_k}\,.$$

We have

$$Card\;p^{-1}(\{A^{I_k}\cap X|_{I_k}\}) \ge Card(\{A^{I_k}\cap X|_{I_k}\})\,,$$

Let fix $\mathbf{U}\subset \bigcup_{n=1}^{\sup(w|_{I_k})}\{0,1\}^n$. We will show that

$$P(\mathbf{U}) = A^{I_k}\cap X|_{I_k}\,.$$

Let us take any $\overline{w}\notin A^{I_k}\cap X|_{I_k}$. From the definition $C_P(w|_{I_k})$ we have $C_P(\overline{w}) \le \sup C_P(w|_{I_k})$. So there is some finite word $(\alpha_1, \alpha_2, ..., \alpha_n) \in \{0, 1\}^n$, $n \le \sup C_P(w|_{I_k})$ such that $P(\alpha_1, \alpha_2, ..., \alpha_n) = \overline{w}$.

Thus $P(\mathbf{U}) = A^{I_k}\cap X|_{I_k}$.

Now we will show that $Card\,\mathbf{U} \le 2^{(a+\varepsilon)|I_k|+1}$.

Indeed, from (1) we have

$$\mathbf{U} = \bigcup_{n=1}^{\sup(w|_{I_k})}\{0,1\}^n \subset \bigcup_{n=1}^{(a+\varepsilon)|I_k|}\{0,1\}^n\,,$$

thus

$$Card\left(\bigcup_{n=1}^{(a+\varepsilon)|I_k|}\{0,1\}^n\right) = \sum_{n=1}^{(a+\varepsilon)|I_k|}\{0,1\}^n =$$
$$= \sum_{n=1}^{(a+\varepsilon)|I_k|} 2^n = 2^{(a+\varepsilon)I_k+1}\,.$$

So we have $Card\,V \le Card\,U \le 2^{(a+\varepsilon)I_k+1}$.

To finish the proof of the theorem we need first some facts about topological entropy.

Theorem 2. Let (X, T) be a symbolic dynamical system. Then

$$h_1(\sigma) = \limsup_{k\to\infty}\frac{1}{|I_k|}\log_2 A_k\,,$$

where $A_k = Card\left|\{w|_{I_k} : w\in X\}\right|$ [4].

From Theorem 2 and (2) we have:

$$A_k \le 2^{(a+\varepsilon)|I_k|+1}\,,$$

and then

$$\limsup_{k\to\infty}\frac{1}{|I_k|}\log A_k \le \limsup_{k\to\infty}\frac{1}{(I_k)}\log 2^{(a+\varepsilon)|I_k|}\,,$$
$$h_t(T) \le a+\varepsilon\,.$$

Hence $h_t(T) \le C_P(X)$.

Now we will prove the inverse inequality. Let $h_t(T) \le b$. Then for $\varepsilon > 0$ there exists $n_0 \in IN$ such that $\forall k > n_0$ we can write

$$\frac{1}{|I_k|}\log A_k \le b+\varepsilon\,,\ \log_2 A_k \le (b+\varepsilon)\,|I_k|\,,$$

$$A_k \le 2^{(b+\varepsilon)|I_k|}.$$

Now let us fix some $k > k_0$. For this k we can define some finite program P such that it is defined on the finite word $\alpha \in \{0,1\}^{\{(b+\varepsilon)|I_k|\}+2}$ and can give us all the finite restriction of the space X on I_k.

Now we will continue with the program P in the following way:

One will divide the big cube I_{km} into $\frac{|I_{km}|}{|I_k|}$ domains every part of which is equal to I_k and now consider the program P on each domain of the big cube. Certainly this program P will be defined on the $\{0, 1\}$ words of length not bigger than

$$(b+\varepsilon)|I_k|\frac{|I_{km}|}{|I_k|} = (b+\varepsilon)|I_{km}|,$$

thus the complexity of the space X relatively to this program P is not bigger than $(b+\varepsilon)$.

Because of that, the complexity of an arbitrary asymptotically optimal program P will not than be bigger than b.

The proof is complete.

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