ISSN 1392 – 124X INFORMATION TECHNOLOGY AND CONTROL, 2007, Vol. 36, No. 1

ON THE CUMULATIVE IDLETIME IN A MODEL OF THE MESSAGE SWITCHING SYSTEM

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Abstract. The purpose of this research in the queueing theory is the theorem about the law of the iterated logarithm in multiphase queueing systems and its application to the mathematical model of the message switching system. First we proof the law of the iterated logarithm for the cumulative idle time of a customer. Then we present an application of the proved theorem for the model of the message switching system.

Key words: mathematical models of technical systems, performance evaluation, queueing theory, multiphase queueing systems, a law of the iterated logarithm, cumulative idle time of a customer.

1. Introduction

At first, the law of the iterated logarithm is considered by investigating the cumulative idle time of a customer in multiphase queueing systems.

Interest in the field of multiphase queueing systems has been stimulated by the theoretical values of the results as well as by their possible applications in information and computing systems, communication networks, and automated technological processes (see, for example, [20]). The methods of investigation of single-phase queueing systems are considered in [2], [3], etc. The asymptotic analysis of models of queueing systems in heavy traffic is of special interest (see, for example, [9], [10], [4], [5], etc.). The papers [11], [18] and others desribed the beginning of the investigation of diffusion approximation to queueing networks. Intermediate models - multiphase queueing systems - are considered seldom due to serious technical difficulties (see, for example, the book [7]). The works on cumulative idle time for the multiphase queueing systems and open Jackson networks in heavy traffic are also scarce. In one of the first papers of this kind [16], Pike used numerical methods to study values of the mean of the cumulative idle time in single-server queues. Takacs [22] obtained limit theorems for the cumulative idle time in the systems $GI/G/1$ and$M/G/1$. MIlch and Waggoner [12] presented expressions for the cumulative idle time of a server in the $GI/G/1$ system. Ridel [19] found the Laplace transform of the distribution of the cumulative idle time in a finite time interval for the $GI/G/1$ system. Kella [8] conceived the Laplace transform of the expected cumulative idle time in an $M/G/1$ queue. Puhalskii [17] considered the moderate-deviation behaviour of the cumulative idle time with single-server queues. These results complement the existing results on the heavy traffic behaviour of this process. Whitt [23] established functional central limit theorems for a cumulative idle time process in a fluid queue. These limit processes have discontinuous sample paths (e.g., to be a non-Brownian stable process, or a more general Levy process).

Let the cumulative idle time of a customer in the phases of a queueing system be unrestricted, the principle of service being “ffirst come, first served". All the random variables studied are defined on one basic probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

We present some definitions in the theory of metric spaces (see, for example, [1]).

Let *C* be a metric space consisting of real continuous functions in [0, 1] with a uniform metric

$$\rho(x, y) = \sup_{0 \le t \le 1} |x(t) - y(t)|, \; x, y \in C.$$

Let D be a space of all real-valued right-continuous functions in [0,1] having left limits and endowed with the Skorokhod topology induced by the metric d (under which D is complete and separable). Also, note that $d(x, y) <= \rho(x, y)$ for $x, y \in D$.

In this paper, we will constantly use an analog of the theorem on converging together (see, for example, [6]):

Theorem 1. *Let* $\varepsilon > 0$ *and* $\boldsymbol{X}_n, \ \boldsymbol{Y}_n, \ \boldsymbol{X} \in D$.
If $\boldsymbol{P}\left(\overline{\lim\limits_{n\to\infty}}\, d(\boldsymbol{X}_n, \boldsymbol{X}) > \varepsilon\right) = 0$
and $\boldsymbol{P}\left(\overline{\lim\limits_{n\to\infty}}\, d(\boldsymbol{X}_n, \boldsymbol{Y}_n) > \varepsilon\right) = 0, \qquad (1)$
then $\boldsymbol{P}\left(\overline{\lim\limits_{n\to\infty}}\, d(\boldsymbol{Y}_n, \boldsymbol{X}) > \varepsilon\right) = 0.$

2. Statement of the problem

We investigate here a k-phase queue (i.e., after a customer has been served in the j-th phase of the queue, he is routed to the $j+1$-th phase of the queue, and, after the service in the k-th phase of the queue, he leaves the queue). Let us denote by t_n the time of arrival of the n-th customer; by $S_n^{(j)}$– the service time of the n-th customer in the j-th phase; $z_n = t_{n+1} - t_n$; and by $\tau_{j,n+j}$ - departure of the n-th customer from the j-th phase of the queue, $j = 1, 2, \cdots, k$.

Let interarrival times (z_n) at the multiphase queueing system and service times $(S_n^{(j)})$ in each phase of the queue for $j = 1, 2, \cdots, k$ be mutually independent identically distributed random variables.

Next, denote by $BI_{j,n}$ the idle time of the n-th customer in the j-th phase of the multiphase queue; $\hat{F}_{j,n} = \sum_{l=1}^{n} BI_{j,l}$ stands for a cummulative idle time of the n-th customer in the j-th phase of the multiphase queue, $j = 1, 2, \ldots, k$.

When $j = 1, 2, \ldots, k$, let

$$\delta_{j,n} = \begin{cases} S_{n-(j-1)}^{(j)} - z_n, & \text{if } n >= k \\ 0, & \text{if } n < k. \end{cases}$$

Let us denote $S_{j,n} = \sum_{l=1}^{n-1} \delta_{j,l}, \ \ S_{0,n} \equiv 0,$ $\hat{S}_{j,n} = S_{j-1,n} - S_{j,n}, \ x_{j,n} = \tau_{j,n} - t_n, \ x_{0,n} \equiv 0,$ $\hat{x}_{j,n+1} = x_{j,n} - \delta_{j,n+1}, \ \hat{x}_{0,n} \equiv 0, \ z_{j,n} = \hat{x}_{j,n} - S_{j,n}, \ \alpha_j = M\delta_{j,n}, \ \alpha_0 \equiv 0, \ Dz_n = \sigma_0^2, \ \ DS_n^{(j)} = \sigma_j^2, \ \tilde{\sigma}_j^2 = \sigma_0^2 + \sigma_j^2, \ S_n^{(0)} = z_n, \ j = 1, 2, \ldots, k,$ $[x]$ as the integer part of number x.

We assume that the following conditions are fulfilled:

there exists a constant $\gamma > 0$ such that

$$\sup_{n>=1} M|S_n^{(j)}|^{4+\gamma} < \infty, \tag{2}$$

$j = 0, 1, 2, \ldots, k$ and

$$\alpha_k < \alpha_{k-1} < \cdots < \alpha_1 < 0. \tag{3}$$

In this paper, we mostly use the equations presented in [13]:

$$\begin{aligned} &\hat{x}_{j,n} = \max_{0<=l<=n} (\hat{x}_{j-1,l} - S_{j,l}) + S_{j,n}, \\ &\hat{x}_{0,n} \equiv 0, \ n >= k, \ j = 1, 2, \ldots, k. \end{aligned} \tag{4}$$

3. On the law of the iterated logarithm for the cumulative idle time of a customer

First we investigate the law of the iterated logarithm for the cumulative idle time in multiphase queues.

We prove the following result.

Theorem 2. *If conditions (2) and (3) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}} \frac{\hat{F}_{j,n} - (-\alpha_j)\cdot n}{\tilde{\sigma_j}\cdot a(n)} = 1\right) =$$

$$\boldsymbol{P}\left(\underline{\lim_{n\to\infty}} \frac{\hat{F}_{j,n} - (-\alpha_j)\cdot n}{\tilde{\sigma_j}\cdot a(n)} = -1\right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(n) = \sqrt{2n \ln\ln n}$.

Proof. Consider the following functions from the space **D**

$$\hat{F}_j^n(t) = \frac{\hat{F}_{j,[nt]} - (-\alpha_j)\cdot [nt]}{a(n)},$$

$$\hat{Z}_j^n(t) = \frac{\hat{z}_{j,[nt]} - (-\alpha_j)\cdot [nt]}{a(n)},$$

$$\hat{S}_j^n(t) = \frac{(-S_{j,[nt]}) - (-\alpha_j)\cdot [nt]}{\sqrt{n}},$$

$j = 1, 2, \ldots, k$ and $0 <= t <= 1$.

Using a triangle inequality, we see that, for each fixed $\varepsilon > 0$,

$$\begin{aligned} &\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\hat{F}_j^n, \hat{S}_j^n) > \varepsilon\right) \\ &<= \mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\hat{F}_j^n, \hat{Z}_j^n) > \frac{\varepsilon}{2}\right) \\ &+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\hat{Z}_j^n, \hat{S}_j^n) > \frac{\varepsilon}{2}\right) <= \\ &\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, \rho(\hat{F}_j^n, \hat{Z}_j^n) > \frac{\varepsilon}{2}\right) \\ &+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, \rho(\hat{Z}_j^n, \hat{S}_j^n) > \frac{\varepsilon}{2}\right) = \\ &\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\sup_{0<=t<=1} |F_{j,[nt]} - \hat{z}_{j,[nt]}|}{a(n)} > \frac{\varepsilon}{2}\right) + \\ &\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\sup_{0<=t<=1} |\hat{z}_{j,[nt]} - (-S_{j,[nt]})|}{a(n)} > \frac{\varepsilon}{2}\right) \\ &= \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max_{0<=l<=n} |F_{j,l} - \hat{z}_{j,l}|}{a(n)} > \frac{\varepsilon}{2}\right) \end{aligned}$$

$$+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0<=l<=n}|\hat{z}_{j,l}-(-S_{j,l})|}{a(n)}>\frac{\varepsilon}{2}\right),$$

$j=1,2,\ldots,k.$

Thus, we have for each fixed $\varepsilon>0$,

$$\begin{aligned}&\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\hat{F}_j^n,\hat{S}_j^n)>\varepsilon\right)<=\\&\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0<=l<=n}|\hat{F}_{j,l}-\hat{z}_{j,l}|}{a(n)}>\frac{\varepsilon}{2}\right)+\\&\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0<=l<=n}|\hat{z}_{j,l}-(-S_{j,l})|}{a(n)}>\frac{\varepsilon}{2}\right),\end{aligned}\tag{5}$$

$j=1,2,\ldots,k.$

It is proved (see [14]) that, if conditions (3) are fulfilled, then, for each fixed $\varepsilon>0$,

$$\mathbf{P}\left(\lim_{n\to\infty}\frac{\max\limits_{0<=l<=n}|\hat{F}_{j,l}-\hat{z}_{j,l}|}{\sqrt{n}}>\varepsilon\right)=0,$$

$j=1,2,\ldots,k.$

Using similar method as in [14], it can be proved that, for each fixed $\varepsilon>0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0<=l<=n}|\hat{F}_{j,l}-\hat{z}_{j,l}|}{a(n)}>\varepsilon\right)=0,\tag{6}$$

$j=1,2,\ldots,k.$

So the first term in (5) converges to zero. We will prove that the second term in (5) converges to zero, too.

Using (4), we see that

$$\begin{aligned}\hat{z}_{j,n}&=\max_{0<=l<=n}(\hat{x}_{j-1,l}-S_{j-1,l}+S_{j-1,l}-S_{j,l})\\&=\max_{0<=l<=n}(\hat{z}_{j-1,l}+S_{j,l}),\quad j=1,2,\ldots,k.\end{aligned}$$

Thus,

$$\hat{z}_{j,n}=\max_{0<=l<=n}(\hat{z}_{j-1,l}+S_{j,l}),\tag{7}$$

$j=1,2,\ldots,k,\quad z_{0,\cdot}\equiv 0.$

Also, we see that

$$\begin{aligned}&\hat{z}_{j,n}-\sum_{i=1}^{j}\hat{S}_{i,n}>=\hat{z}_{j-1,n}+\hat{S}_{j,n}-\sum_{i=1}^{j}\hat{S}_{i,n}\\&=\hat{z}_{j-1,n}-\sum_{i=1}^{j-1}\hat{S}_{i,n}>=\cdots>=\\&\hat{z}_{1,n}-\hat{S}_{1,n}=\max_{0<=l<=n}(\hat{S}_{1,n})-\hat{S}_{1,n}>=0.\end{aligned}$$

So,

$$\hat{z}_{j,n}-\sum_{i=1}^{j}\hat{S}_{i,n}>=0,\quad j=1,2,\ldots,k.\tag{8}$$

But

$$\begin{aligned}\hat{z}_{j,n}&<=\max_{0<=l<=n}(\hat{z}_{j-1,l})+\max_{0<=l<=n}\hat{S}_{j,l}\\&=\hat{z}_{j-1,n}+\max_{0<=l<=n}\hat{S}_{j,l}<=\cdots\\&<=\sum_{i=1}^{j}\{\max_{0<=l<=n}\hat{S}_{i,l}\}.\end{aligned}$$

It follows that

$$\hat{z}_{j,n}<=\sum_{i=1}^{j}\{\max_{0<=l<=n}\hat{S}_{i,l}\},\quad j=1,2,\ldots,k.\tag{9}$$

Using (8) and (9) we get that

$$\begin{aligned}0&<=\hat{z}_{j,n}-\sum_{i=1}^{j}\hat{S}_{i,n}\\&<=\sum_{i=1}^{j}\{\max_{0<=l<=n}\hat{S}_{i,l}-\hat{S}_{i,n}\},\end{aligned}\tag{10}$$

$j=1,2,\ldots,k.$

Applying (9) we achieve for each fixed $\varepsilon>0$,

$$\begin{aligned}&\mathbf{P}\left(\frac{\max\limits_{0<=l<=n}|\hat{z}_{j,l}-\sum\limits_{i=1}^{j}\hat{S}_{j,l}|}{a(n)}>\varepsilon\right)\\&=\mathbf{P}\left(\frac{\max\limits_{0<=l<=n}(\hat{z}_{j,l}-\sum\limits_{i=1}^{j}\hat{S}_{j,l})}{a(n)}>\varepsilon\right)<=\end{aligned}\tag{11}$$

$$\begin{aligned}&\mathbf{P}\left(\frac{\sum\limits_{i=1}^{j}\max\limits_{0<=l<=n}\{\max\limits_{0<=m<=l}\hat{S}_{i,m}-\hat{S}_{i,l}\}}{a(n)}>\varepsilon\right)\\&<=\mathbf{P}\left(\frac{\sum\limits_{i=1}^{k}\max\limits_{0<=l<=n}\{\max\limits_{0<=m<=l}\hat{S}_{i,m}-\hat{S}_{i,l}\}}{a(n)}>\varepsilon\right)\end{aligned}$$

$$<= \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max\limits_{0<=l<=n}\{\max\limits_{0<=m<=l}\hat{S}_{i,m}-\hat{S}_{i,l}\}}{a(n)} > \frac{\varepsilon}{k}\right)$$

$$<= \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max\limits_{0<=l<=n}\{\max\limits_{0<=m<=l}\hat{S}_{i,m}-\hat{S}_{i,l}\}}{a(n)} > \frac{\varepsilon}{k}\right)$$

$$= \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max\limits_{0<=l<=n}\{\max\limits_{0<=m<=l}(-\hat{S}_{i,l-m})\}}{a(n)} > \frac{\varepsilon}{k}\right)$$

$$= \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max\limits_{0<=l<=n}\{\max\limits_{0<=m<=l}(-\hat{S}_{i,m})\}}{a(n)} > \frac{\varepsilon}{k}\right)$$

$$<= \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max\limits_{0<=l<=n}(-\hat{S}_{i,l})}{a(n)} > \frac{\varepsilon}{k}\right), \ j=1,2,\ldots,k.$$

Thus, we have that for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\frac{\max\limits_{0<=l<=n}|\hat{z}_{j,l}-\sum\limits_{i=1}^{j}\hat{S}_{j,l}|}{a(n)} > \varepsilon\right) <= \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max\limits_{0<=l<=n}(-\hat{S}_{i,l})}{a(n)} > \frac{\varepsilon}{k}\right), \quad j=1,2,\ldots,k. \tag{12}$$

Note (see, for example, [14]) that for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0<=l<=n}(-\hat{S}_{i,l})}{a(n)} > \varepsilon\right) = 0, \tag{13}$$

$j = 1, 2, \ldots, k$, if conditions (3) are fulfilled.

Using relation

$$\sum_{i=1}^{k} \hat{S}_{i,n} = -S_{j,n}, \ \ j=1,2,\ldots,k$$

and (12) - (13) we obtain that for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0<=l<=n}|\hat{z}_{j,l}-(-S_{j,l})|}{a(n)} > \varepsilon\right) = 0, \quad j=1,2,\ldots,k. \tag{14}$$

Using the theorem on the law of the iterated logarithm for random functions $\hat{S}_j^n(t), \ \ j=1,2,\ldots,k$ (see, for example, [21]) we achieve that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{(-S_{j,n})-(-\alpha_j)\cdot n}{\tilde{\sigma}_j\cdot a(n)} = 1\right) = 1 \text{ and } \mathbf{P}\left(\underline{\lim_{n\to\infty}} \frac{(-S_{j,n})-(-\alpha_j)\cdot n}{\tilde{\sigma}_j\cdot a(n)} = -1\right) = 1, \quad j=1,2,\ldots,k. \tag{15}$$

Thus, applying (1), (5), (6), (14) and (15) we obtain that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\hat{F}_{j,n}-(-\alpha_j)\cdot n}{\tilde{\sigma}_j\cdot a(n)} = 1\right) = 1 \text{ and } \mathbf{P}\left(\underline{\lim_{n\to\infty}} \frac{\hat{F}_{j,n}-(-\alpha_j)\cdot n}{\tilde{\sigma}_j\cdot a(n)} = -1\right) = 1, \quad j=1,2,\ldots,k. \tag{16}$$

The proof of Theorem 2 is complete. □

4. On the model of switching facility

In this section, we will present an application of the proved theorem - a mathematical model of message switching system.

As noted in the introduction, multiphase queueing systems are of special interest both in theory and in practical applications. Such systems consist of several service nodes, and each arriving customer is served at each of the consecutively located node (frequently called phases). A typical example is provided by queueing systems with identical service. Such systems are very important in applications, especially to message switching systems. In fact, in many comunication systems the transmission times of the customers do not vary in the delivery process.

So, we investigate a message switching system which consists of k phases and in which $S_n^j = S_n, \ \ j = 1, \ 2, \ldots, k$ (the service process is identical in phases of the system).

Let

$$\delta_n = \begin{cases} S_{n-k} - z_n, & \text{if } n >= k \\ 0, & \text{if } n < k. \end{cases}$$

Also, let us denote $\alpha = M\delta_n, Dz_n = \sigma_0^2, \ DS_n = \sigma^2, \ \tilde{\sigma}^2 = \sigma_0^2 + \sigma^2, \hat{F}_{j,n} = \sum\limits_{l=1}^{n} BI_{j,l}, j = 1,2,\ldots,k.$

We assume that the following conditions are fulfilled:

there exists a constant $\gamma > 0$ such that

$$\sup_{n>=1} M|S_n|^{4+\gamma} < \infty \tag{17}$$

and

$$\alpha < 0. \tag{18}$$

Similarly as in the proof of Theorem 3.1, we present the following theorem on the law of the iterated logarithm for the cumulative idle time of a data packet in message switching systems.

Theorem 3. *If conditions (17) and (18) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}} \frac{\hat{F}_{j,n} - (-\alpha)\cdot n}{\tilde{\sigma}\cdot a(n)} = 1\right)$$
$$= \boldsymbol{P}\left(\underline{\lim_{n\to\infty}} \frac{\hat{F}_{j,n} - (-\alpha)\cdot n}{\tilde{\sigma}\cdot a(n)} = -1\right) = 1,$$

$j = 1, 2, \ldots, k.$

We see that the cumulative idle time of data packet is the same in all the phases of system.

5. A numerical example

We see that Theorem 3 implies that for fixed $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for every $n >= n(\varepsilon)$, with probability one

$$\begin{aligned}(1-\varepsilon)\cdot\tilde{\sigma}\cdot a(n) - \alpha\cdot n <= \hat{F}_{j,n}\\ <= (1+\varepsilon)\cdot\tilde{\sigma}\cdot a(n) - \alpha\cdot n,\end{aligned} \tag{19}$$

where $a(n) = \sqrt{2n\ln\ln n}$, $\alpha = M(S_n - z_n) < 0$, $\tilde{\sigma}^2 = Dz_n + DS_n$, $j = 1, 2, \ldots, k$.

From this we can obtain

$$\begin{aligned}&(1-\varepsilon)\cdot\tilde{\sigma}\cdot a(n) - \alpha\cdot n <= \hat{F}_{j,n}\\ &<= (1+\varepsilon)\cdot\tilde{\sigma}\cdot a(n) - \alpha\cdot n,\\ &|M(\hat{F}_{j,n} - (-\alpha)\cdot n) - \{(1-\varepsilon)\cdot\tilde{\sigma}\cdot a(n)\}|\\ &<= 2\cdot\varepsilon\cdot\tilde{\sigma}\cdot a(n),\\ &|M\left(\frac{\hat{F}_{j,n} - (-\alpha)\cdot n)}{\tilde{\sigma}\cdot a(n)}\right) - (1+\varepsilon)|\\ &<= 2\cdot\varepsilon,\ j = 1, 2, \ldots, k.\end{aligned} \tag{20}$$

So from (20) we can get

$$M\hat{F}_{j,n} \sim (-\alpha)\cdot n + (1+\varepsilon)\cdot\tilde{\sigma}\cdot a(n), \tag{21}$$

$j = 1, 2, \ldots, k$. $M\hat{F}_{j,n}$ is average cumulative idle time of the n-th message (time, which system is waiting for processing message until the n-th message arrival to the system).

We see from (21) that $M\hat{F}_{j,n}$ consists of linear function and nonlinear slowly increasing function $(1+\varepsilon)\cdot\tilde{\sigma}\cdot a(n)$, $j = 1, 2, \ldots, k$.

Now we present a technical example from the computer network practice. Assume that messages arrive at the computer v_1 at the rate λ of 20 per hour during business hours. These messages are served at a rate μ of 25 per hour in the computer v_1. After service in the computer v_1 messages arrive at the second computer v_2. Also we note that messages are served at a rate μ of 25 per hour in the computer v_2. So, messages are served in computers v_1, v_2,…,v_k, and after messages have been served in computer v_k, they leave computer network.

So, $Mz_n = 1/\lambda = 1/20 = 0.05$, $MS_n = 1/\mu = 1/25 = 0.04$, $\alpha = 0.04 - 0.05 = -0.01 < 0, Dz_n = 1/\lambda = 1/20 = 0.05$, $DS_n = 1/\mu = 1/25 = 0.04$, $\tilde{\sigma}^2 = 41/10^4$, $\tilde{\sigma} \sim 0.064$, $\varepsilon = 0.001, n >= 100$.

Thus,

$$\begin{aligned}&M\hat{F}_{j,n} \sim (-\alpha)\cdot n + (1+\varepsilon)\cdot\tilde{\sigma}\cdot a(n) =\\ &(0.01)\cdot n + (0.064)\cdot a(n),\ \ j = 1, 2, \ldots, k.\end{aligned} \tag{22}$$

From (22) we get

$$\begin{aligned}&\frac{M\hat{F}_{j,n}}{n} = (0.01) + (0.064)\cdot\sqrt{\frac{2\ln\ln n}{n}},\\ &j = 1, 2, \ldots, k.\end{aligned} \tag{23}$$

Now we present figure the values of $\frac{M\hat{F}_{j,n}}{n}$, $j = 1, 2, \ldots, k$, when $100 <= n <= 1000$, $\varepsilon = 0.001$ (see (23) and Table 1).

Time n	$\frac{M\hat{F}_{j,n}}{n}$, $j = 1, 2, \ldots, k$
100	0.02118510415
200	0.01826415546
300	0.01689524794
400	0.01605525217
500	0.01547101209
600	0.01503369681
700	0.01469010032
800	0.01441067288
900	0.01417749453
1000	0.01397897294

Table 1 Summary of computing results.

We see that when $\alpha = -0.01 < 0$, computer network is busy 99 % of this time.

Corollary 5.1. *The average idle time of message system direcly depends on the traffic coefficient α and time n and is the same in all phases of message system.*

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Received August 2006.