STRONGLY SUMMABLE AND STATISTICALLY CONVERGENT FUNCTIONS

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Abstract. Strongly summable sequences and lacunary strongly summable sequences were studied by several authors including [5]. Also statistically convergent sequences and lacunary statistical convergent sequences were studied several authors, including [2],[3],[6],[7].In this paper instead of sequences, by taking real valued functions x,measurable (in the Lebesque sense) in the interval (1;1), we have given definitions of summability, strong summability, lacunary convergence, lacunary strong convergence, strongly almost convergence, statistical convergence and lacunary statistical convergence of these functions. Also we have given some inclusion relations.

1. Introduction

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_\theta = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by $\theta = \{k_r\}$ will be denoted by $I_r = (k_{r-1}, k_r)$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. The space of strongly Cesaro summable sequences is defined by

$[w]=\{x=(x_k)$: there exists l such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_k - l| = 0 \}$$

Definition 1.1. *[1][W] is the space of real valued functions x, measurable (in the Lebesque sense) in the interval* $(1; \infty)$, *for which there is a number l=l(x) such that* $\lim_{T\to\infty} \frac{1}{T}\int_1^T |x(t)-l|\,dt = 0$ *with norm* $\|x\| = \sup_{T\geq 1}\left(\frac{1}{T}\int_1^T |x(t)|\,dt\right)$. *We denote those functions in* $[W]$ *for which* $l = 0$ *by* $[W_0]$.

2. Lacunary strongly convergent sequences

Definition 2.1. *Let* $\theta = \{k_r\}$ *be a lacunary sequence,* N_θ *is the space of real valued function x, measurable (in the Lebesque) in the interval* $(1; \infty)$, *for which there is a number* $l = l(x)$ *such that*

$\lim_{r\to\infty} \frac{1}{h_r}\int_{k_{r-1}}^{k_r} |x(t)-l|\,dt = 0$ *with norm* $\|x\| = \sup_r\left(\frac{1}{h_r}\int_{k_{r-1}}^{k_r} |x(t)|\,dt\right)$. *We denote those functions in* $[N_\theta]$ *for which* $l = 0$ *by* $[N_\theta^0]$.

Let $\theta = \{k_r\}$ be lacunary sequence. If we define a function $x = x(t)$ by

$$x(t) = \begin{cases} \frac{2}{h_r} sign\beta(t) & \text{if } t \in I_r \\ 0 & \text{otherwise} \end{cases},$$

then $x \in N_\theta^0$.

Theorem 2.2. *For any lacunary sequence* $\theta = \{k_r\}$, $[W] \subseteq N_\theta$ *if and only if* $\lim\inf_r q_r > 1$.

Proof. If $\lim\inf_r q_r > 1$ then we can find $\lambda > 0$ that $1 + \lambda \leq q_r$ for all $r \geq 1$. For $x(t) \in [W_0]$ we can write

$$\frac{1}{h_r}\int_{k_{r-1}}^{k_r} |x(t)|\,dt = \frac{1}{h_r}\int_1^{k_r} |x(t)|\,dt - \frac{1}{h_r}\int_1^{k_{r-1}} |x(t)|\,dt$$
$$= \frac{k_r}{h_r}\left(\frac{1}{k_r}\int_1^{k_r} |x(t)|\,dt\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\int_1^{k_{r-1}} |x(t)|\,dt\right).$$

Since $\frac{k_r}{h_r} \leq \frac{1+\lambda}{\lambda}$, $\frac{k_{r-1}}{h_r} \leq \frac{1}{\lambda}$ and $x(t) \in [W_0]$ we have $x(t) \in N_\theta^0$. The general inclusion $[W] \subseteq N_\theta$ follows by linearity.

Now assume that $\lim\inf_r q_r = 1$. Since θ is a lacunary sequence, we can select a subsequence (k_{rj}) of θ satisfying

$$\frac{k_{r_j}}{k_{r_{j-1}}} < 1 + \frac{1}{j} \text{ and } \frac{k_{r_j-1}}{k_{r_{j-1}}} < j \text{ where } r_j \geq r_{j-1} + 2.$$

Let

$$x(t) = \begin{cases} 1 & \text{if } t \in I_{r_j} \text{ for some } j = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}.$$

Then for any real number l,

$$\frac{1}{h_{r_j}} \int_{k_{r_j-1}}^{k_{r_j}} |x(t) - l|\, dt = |1 - l| \text{ for } j = 1, 2, \dots, \text{ and}$$

$$\frac{1}{h_r} \int_{k_{r-1}}^{k_r} |x(t)|\, dt = |l| \text{ for } r \neq r_j.$$

Hence $x \notin N_\theta$. If u is any sufficiently large integer we can find unique j for $k_{r_j-1} < u < k_{r_{j+1}-1}$ and write

$$\frac{1}{u} \int_1^u |x(t)|\, dt \leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j-1}} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

As $u \to \infty$ it follows that also $j \to \infty$ and $x \in [W_0]$.

We can prove the following theorem by using similar techniques to Lemma 2.2 of [2].

Theorem 2.3. *For any lacunary sequence* $\theta = \{k_r\}$, $[W] = N_\theta$ *if and only if* $\limsup_r q_r < \infty$.

Combining Theorem 2.2 and Theorem 2.3 we get

Theorem 2.4. *For any lacunary sequence* $\theta = \{k_r\}$, $[W] = N_\theta$ *if and only if* $1 < \liminf_r q_r < \limsup_r q_r < \infty$.

3. Statistical and lacunary statistical convergence

Definition 3.1. *A real valued function x measurable (in the Lebesque sence) in the interval* $(1, \infty)$, *is said to be statistically convergent to the number* $l = l(x)$ *if for every* $\varepsilon > 0$,

$$\lim_{T\to\infty} \frac{1}{T} |\{t \leq T : |x(t) - l| \geq \varepsilon\}| = 0.$$

where the vertical bars indicate the Lebesque measure of the enclosed set. In this case we write $x(t) \to l(F)$ *and we define*

$F = \{x(t) : \text{for some } l,\ x(t) \to l(F)\}$.

Definition 3.2. *Let* $\theta = \{k_r\}$ *be a lacunary sequence. A real valued function x, measurable (in the Lebesque sence) in the interval* $(1; \infty)$, *is said to be lacunary statistically convergent to the number* $l = l(x)$ *if for every* $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{t \in I_r : |x(t) - l| \geq \varepsilon\}| = 0.$$

In this case we write $x(t) \to l(F_\theta)$ *and we define*

$F = \{x(t) : \text{for some } l,\ x(t) \to l(F_\theta)\}$.

We can prove the following theorem by using similar techniques to Lemmas 2.1, 2.2 and Theorem 2.1of [4].

Theorem 3.3. *Let* $\theta = \{k_r\}$ *be a lacunary sequence. Then* (*i*) $F_\theta \subset F$ *if and only if* $\limsup_r q_r < \infty$, (*ii*) $F_\theta \subset F$ *if and only if* $\liminf_r q_r > 1$, (*iii*) $F = F_\theta$ *if and only if* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$.

We can prove the following theorem by using similar techniques to Theorem 1 of [7].

Theorem 3.4. *Let* $\theta = \{k_r\}$ *be a lacunary sequence. Then* (*i*) $x(t) \to l(N_\theta)$ *implies* $x(t) \to L(F_\theta)$, (*ii*) *If* $x(t)$ *is a bounded function and* $x(t) \to L(F_\theta)$, *then* $x(t) \to l(N_\theta)$, (*iii*) *For bounded function* $x(t)$, $F_\theta = N_\theta$.

4. Strong almost convergence

Definition 4.1. $|\hat{C}|$ *is the space of real valued functions x measurable (in the Lebesque sence) in the interval* $(1, \infty)$, *for which there is a number* $l = l(x)$ *such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_{m+1}^{m+T} |x(t) - l|\, dt = 0 \text{ uniformly in } m = 0, 1, 2, \dots$$

We can prove the following theorem by using techniques to Lemma 3.1 [2].

Theorem 4.2. $|\hat{C}| \subset N_\theta$ *for every lacunary sequence* $\theta = \{k_r\}$.

5. The space W and C_θ

Definition 5.1. *W is the space of real valued functions x measurable (in the Lebesque sence) in the interval* $(1, \infty)$, *for which there is a number* $l = l(x)$ *such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_{m+1}^{m+T} x(t)\, dt = l.$$

If we define

$$x(t) = \begin{cases} 2^n & if\ n < t < n = \frac{1}{2^n},\ n = 1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$$

then $x \in W$ and $l = 1$.

Definition 5.2. *For any lacunary sequence* $\theta = \{k_r\}$, c_θ *is the space of real valued functions x measurable (in the Lebesque sense) in the interval* $(1, \infty)$, *for which there is a number* $l = l(x)$ *such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_{m+1}^{m+T} x(t)\, dt = l$$

Theorem 5.3. $W \subset C_\theta$ *if and only if* $\liminf_r q_r > 1$.

Proof. Suppose that lim $\inf_r q_r > 1$. If $x \in W$, then there exists l such that

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T x(t)dt = l\ .$$

We write

$$\frac{1}{h_r}\int_{k_{r-1}}^{k_r} x(t)dt = \frac{1}{h_r}\left(\int_1^{k_r} x(t)dt - \int_1^{k_{r-1}} x(t)dt\right)$$

$$= \frac{k_r}{h_r}\frac{\int_1^{k_r} x(t)dt}{k_r} - \frac{k_{r-1}}{h_r}\frac{\int_1^{k_{r-1}} x(t)dt}{k_{r-1}} = b_r y_r + m_r g_r\ .$$

where $y_r \to l$, $g_r \to l$ and (b_r) and (m_r) are bounded sequences satisfying $b_r + m_r = 1$ for $r = 1, 2, \ldots$. Then we have that $| b_r y_r + b_r y_r - l| = | b_r(y_r - l) + m_r(g_r - l)| \le | b_r| |y_r - l| + | m_r| |g_r - l| \to 0$ and $x \in c_\theta$.

Suppose that lim $\inf_r q_r > 1$. Then we define a function $x(t)$ as follows:

$$x(t) = \begin{cases} 1 & \text{if } t \in I_{r_j}, \text{for some } j = 1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

This function is in W and not in c_θ.

References

[1] **D. Borwein.** Linear functionals with strong Cesaro Summability. *Journal London Math.Soc.*, 40, 1965, 628-634.

[2] **J.A. Fridy.** On statistical convergence. *Analysis* 5, 1985, 301-313.

[3] **J.A. Fridy, C. Orhan.** Lacunary statistical convergence. *Pacific Journal Math.* 160, 1993, 43-51.

[4] **A.R. Freedman, J.J. Sember, M. Raphael.** Some Cesaro-type summability spaces. *Proc.London Math.Soc.* 37(3), 1978, 508-520.

[5] **I.J. Maddox.** Space of strongly summable sequences. *Quart. J.Math.Oxford* (2)18, 1967, 345-355.

[6] **F. Nuray, W.H Ruckle.** Generalized statistical convergence and convergence free spaces. *J.Math.Anal. appl.* 2, 2000, 513-527.

[7] **E. Savas, F. Nuray.** On ¾-statistically convergence and lacunary ¾ – statistically convergence. *Math.Slovaca* 43, 1993, 309-315.