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NUMERICAL IMPLEMENTATION OF THE ABEL TRANSFORM FOR THE CONSTRUCTION OF DIGITAL FLUID HOLOGRAPHIC IMAGES

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Abstract. Numerical implementation of the Abel transform for the construction of digital fluid holographic images is analyzed in this paper. Volumetric strains are obtained from the numerical calculations using the displacement formulation by the method of finite elements. Then the field of volumetric strain is obtained by using the procedure of conjugate approximation. Numerical procedure is developed for the calculation of the Abel transform on the basis of the right rectangle numerical integration rule and thus the digital holographic image is constructed. The obtained digital holographic images are used in the hybrid experimental – numerical procedure for the determination of the correlation with the experimental holographic images.

Keywords: digital image, computer graphics, Abel transform, numerical integration, finite elements.

1. Introduction

Development of hybrid numerical – experimental techniques is an important method of analysis used for interpretation and validation of experimental results [1]. Numerical implementation of the Abel transform for the construction of digital fluid holographic images is analyzed in this paper.

Laser holography is a powerful experimental technique for analysis of high frequency vibrations of the fluid, especially when the amplitudes of those vibrations are relatively small [2, 3, 4]. The volumetric strains are obtained from the numerical calculations using the displacement formulation by the method of finite elements [5, 6]. Then the field of volumetric strain is obtained by using the procedure of conjugate approximation [7].

A schematic diagram of the experimental setup is shown in Figure 1. A beam from a laser is split into two beams by a beam splitter forming the object and the reference beams. The object beam is collimated by a spherical lens 1 and a parabolic mirror 1 before passing through the investigated fluid. After passing it the diameter of the beam is reduced by the combination of the parabolic mirror 2 and the spherical lens 2. They image the investigated fluid onto the holographic film. The reference beam passes through the mirrors 2 and 3 and then is expanded by a lens 3. It illuminates the holographic film interfering with the object beam. x, y, z denote the orthogonal Cartesian axes of coordinates (y axis is perpendicular to the drawing and thus is not indicated in the figure).

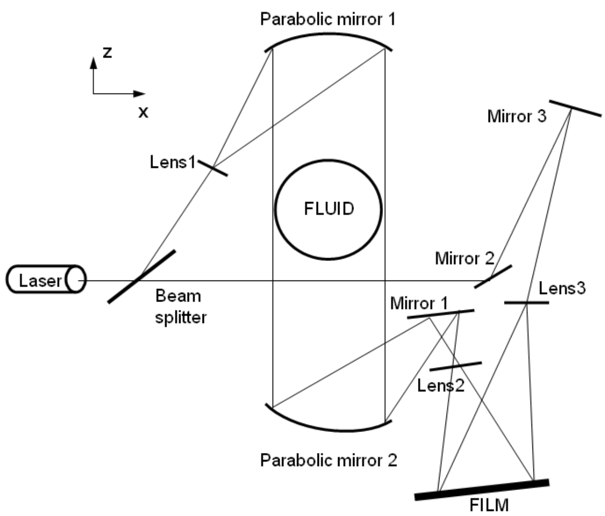

Figure 1. Schematic diagram of the experimental setup.

After performing the numerical calculation of the Abel transform the digital holographic image is constructed. It is compared with the experimental holographic image and thus the hybrid experimental – numerical procedure is implemented.

2. Model of the system

It is assumed that the angular frequency of excitation coincides with the eigenfrequency of the appropriate eigenmode and thus the eigenproblem is analyzed.

The mass matrix of the fluid is:

$$[M]=\int [N]^T \rho [N] 2\pi x dxdy \,, \qquad (1)$$

where ρ is the density of the fluid in the status of equilibrium, x is the radial co-ordinate; y is the vertical co-ordinate (the axis of symmetry); $[N]$ is the matrix of the shape functions defined by the following relationship:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{\delta\} \,, \qquad (2)$$

where u, v denote the displacements of the fluid in the directions of the axes x and y in the domain of the appropriate finite element; $\{\delta\}$ is the nodal displacement vector. Explicitly:

$$[N]=\begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix}, \qquad (3)$$

where N_i are the shape functions of the analyzed finite element. The stiffness matrix of the fluid is:

$$[K]=\int \left([\overline{B}]^T \rho c^2 [\overline{B}] + [\widetilde{B}]^T \lambda [\widetilde{B}] \right) 2\pi x dxdy \,, \qquad (4)$$

where c is the speed of sound in the fluid; λ is the penalty parameter for the condition of irrotationality. The matrix [$\overline{B}$] relating the volumetric strain with the displacements is defined from:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{x}=[\overline{B}]\{\delta\}. \qquad (5)$$

Explicitly:

$$[\overline{B}]= \begin{bmatrix} \frac{\partial N_1}{\partial x}+\frac{N_1}{x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x}+\frac{N_2}{x} & \frac{\partial N_2}{\partial y} & \cdots \end{bmatrix}. \qquad (6)$$

The matrix [$\widetilde{B}$] is used to characterize the rotation and is defined from:

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=[\widetilde{B}]\{\delta\}. \qquad (7)$$

Thus:

$$[\widetilde{B}]= \begin{bmatrix} \frac{\partial N_1}{\partial y} & -\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & -\frac{\partial N_2}{\partial x} & \cdots \end{bmatrix}. \qquad (8)$$

Conventional finite element analysis techniques are based on the approximation of nodal displacements via the shape functions. But volumetric strains (not nodal displacements) are involved in the relationships governing the intensity of illumination in the hologram. Thus the field of volumetric strain is calculated using the procedure of conjugate approximation.

First, the volumetric strains at the points of numerical integration of the finite element are calculated in the usual way:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{x}=[\overline{B}]\{\delta_0\}, \qquad (9)$$

where $\{\delta_0\}$ is the vector of nodal displacements of the eigenmode.

Thus the following system of linear algebraic equations for the determination of nodal volumetric strains is solved:

$$\begin{aligned} &\iint [\hat{N}]^T [\hat{N}] 2\pi x dxdy \cdot \{\delta_v\} = \\ &= \iint [\hat{N}]^T \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{x} \right) 2\pi x dxdy, \end{aligned} \qquad (10)$$

where $\{\delta_v\}$ is the vector of nodal values of $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{x}$ (the eigenmode of volumetric strains); $\lfloor \hat{N} \rfloor$ is the row of the shape functions of the finite element:

$$\lfloor \hat{N} \rfloor = [N_1 \quad N_2 \quad ...]. \qquad (11)$$

The obtained volumetric strains are used in the procedure of numerical calculation of the Abel transform.

3. Numerical implementation of the Abel transform

The notation used in the Abel transform is explained in detail in Figure 2, where r is the radial coordinate.

The Abel transform has the form:

$$\Phi(x,y)=2\int_x^{\infty} \varepsilon_v(r,y)\frac{rdr}{\sqrt{r^2-x^2}}\,. \qquad (12)$$

Numerical implementation of the calculation of the Abel transform is performed in three steps:

a) by using the procedure described in [8] and assuming that the projection plane coincides with the plane z=0, the column number and the row number of a pixel with the corresponding value of the volumetric strain is obtained for all the analyzed values of the local coordinates for each finite element;
b) for each scan line of the digital image, the averaged volumetric strain for each pixel of the scan line is obtained;

c) for each scan line of the digital image, by using the right rectangle numerical integration rule the Abel transform is calculated and then the digital holographic image is produced.

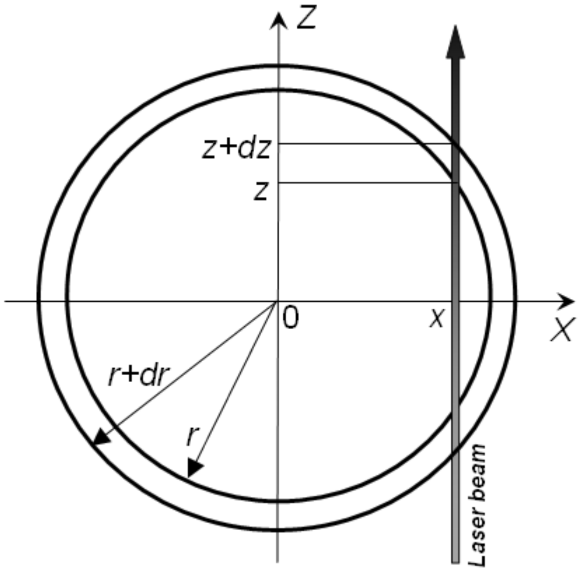

Figure 2. The Abel transform

4. Numerical results

Axi-symmetric problem in a rectangular domain is analyzed. The right boundary is rigid and the displacements normal to it are set to zero. The left boundary is the axis of symmetry and the displacements normal to it are set to zero also. Periodic boundary conditions on the upper and lower boundaries are assumed: the displacements on those surfaces for the same values of the radial coordinate are assumed mutually equal.

The digital stroboscopic and time averaged holographic images of the second eigenmode are presented in Figure 3. The digital stroboscopic and time averaged holographic images of the fourth eigenmode are presented in Figure 4. The digital stroboscopic and time averaged holographic images of the fifth eigenmode are presented in Figure 5.

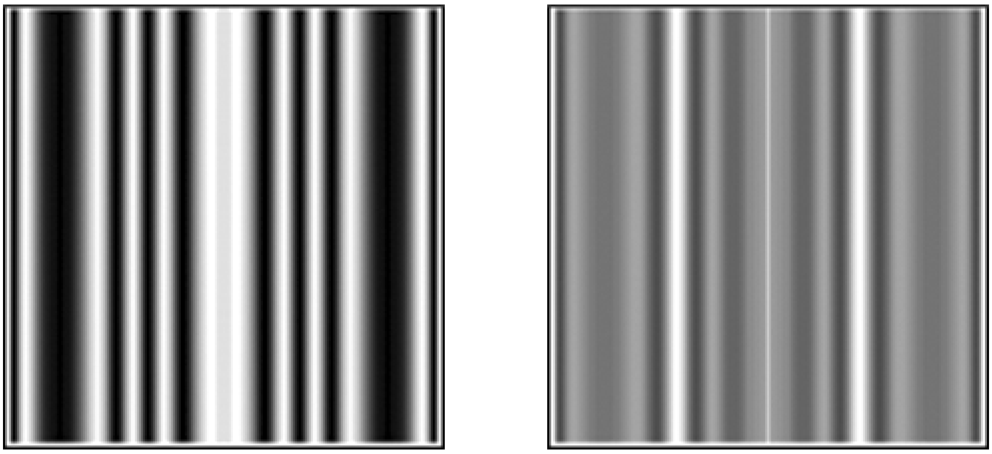

Figure 3. Digital holographic images (stroboscopic and time averaged) of the second eigenmode

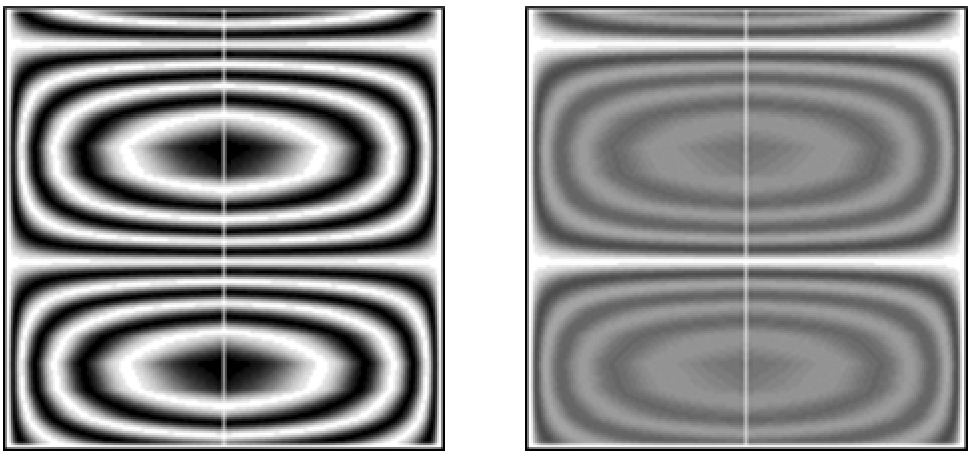

Figure 4. Digital holographic images (stroboscopic and time averaged) of the third eigenmode

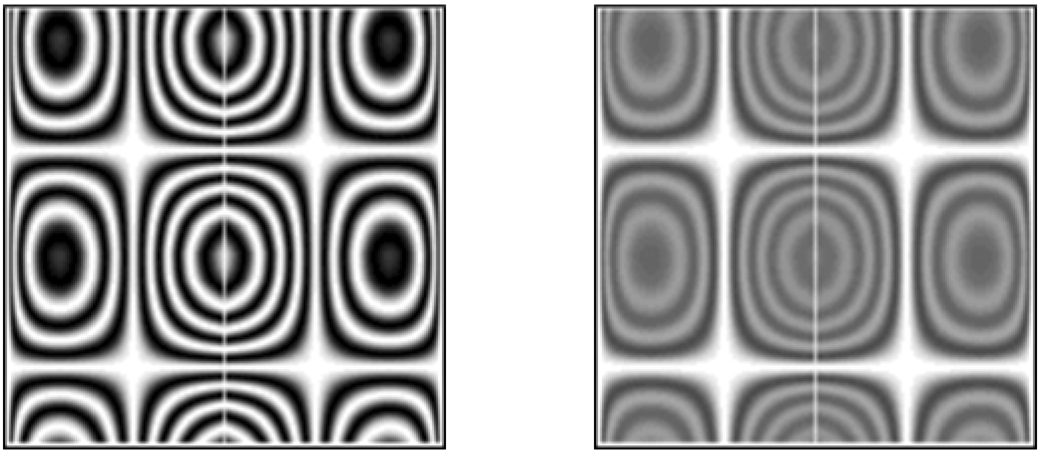

Figure 5. Digital holographic images (stroboscopic and time averaged) of the fourth eigenmode

5. Conclusions

The numerical procedure for the calculation of nodal values of volumetric strain corresponding to the eigenmodes of the axi-symmetric problem of vibrations of the fluid is used as the first stage of the analysis.

The second stage of the analysis is based on the obtained nodal values of the amplitudes of the volumetric strain which are used in the construction of the digital holographic images. It is based on the application of the right rectangle numerical integration rule.

The obtained digital holographic images are used in the hybrid experimental – numerical procedure for the determination of the correlation with the experimental holographic images.

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