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Quantized Feedback Control Over Packet Dropout Communication Channels

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Abstract. This paper investigates quantized feedback control problems for linear time-invariant control systems, where the sensors and controllers are geographically separated and connected via noisy, bandwidth-limited digital communication channels. The packet dropout process of the channel is modeled as a time-homogeneous Markov process. An adaptive differential coding strategy and a predictive control policy are implemented to achieve the minimum data rate of the channel for mean square stabilization of the unstable plant. In particular, it is shown that a sufficient condition on mean square stabilization of the system with disturbances is that the data rate is more than the lower bound given in our results. The sufficient condition decomposes into two terms: a condition on the data rate and a condition on the transition probabilities of the Markov chain. An illustrative example is given to demonstrate the effectiveness of the proposed method.

Keywords: quantized feedback control, data rates, mean square stabilization, networked control systems.

1. Introduction

Networked control systems (NCSs) operate subject to communication constraints, which arise from the characteristics of practical channels, such as data rate limits, random time delays and packet dropout. Feedback control under communication constraints has been the focus of much recent research [28].

Issues of the type discussed are motivated by several pieces of work in the recent literature. The research on the interplay among coding, estimation, and control was initiated by [1]. A high-water mark in the study of quantized feedback using data rate limited feedback channels is known as the data rate theorem that states the larger the magnitude of the unstable poles, the larger the required data rate through the feedback loop. The intuitively appealing result was proved [2-5], indicating that it quantifies a fundamental relationship between unstable physical systems and the rate at which information must be processed in order to stably control them. When the feedback channel capacity is near the data rate limit, control designs typically exhibit chaotic instabilities. This result was generalized to different notions of stabilization and system models, and was also extended to multi-dimensional systems [6-8]. The research on Gaussian linear systems was addressed in [9-11]. Information theory was employed in control systems as a powerful conceptual aid, which extended existing fundamental limitations of feedback systems, and was used to derive necessary and sufficient conditions for robust stabilization of uncertain linear systems, Markov jump linear systems and unstructured uncertain systems [12-16]. The decentralized control schemes were addressed in [17]. The result on continuous-time linear Gaussian systems was derived in [18]. The result on time-varying communication channel was derived in [19]. A switched-adaptive quantization technique using μ-law quantizers was addressed in [29]. Liu et al. [30] investigates quantized control problems for linear time-invariant systems over a noiseless communication network. The survey papers [20] and [21] gave a historical and technical account of the various formulations.

Hespanha et al. [22] addressed stabilization of systems over communication networks of infinite bandwidth subject to packet losses. Sinopoli et al. [23] modeled the packet dropout rate above which the mean state estimation error covariance will diverge. Elia et al. [24] extended the results and addressed the mean square stability of an SISO plant. You et al. [25] investigated the minimum data rate for mean square stabilization of linear systems over a lossy digital channel, where the packet dropout process is modeled as a Markov chain.

Motivated by their work, we address quantized feedback control problems for linear systems over a noisy, bandwidth-limited digital channel. In particular, it is shown in our results that there exists a quantization, coding, and control scheme to stabilize the unstable plant if the data rate is more than the lower bound given in our results. The sufficient condition decomposes into two terms: a condition on the data rate and a condition on the transition probabilities of the Markov chain. Our work here differs in that we employ an adaptive differential coding strategy and a predictive control policy to stabilize the unstable plant, and present sufficient conditions on the data rate for mean square stabilization, which is less conservative than those of the literature.

The rest of the paper is organized as follows. In Section 2, the problem formulation is presented. Section 3 deals with quantized feedback control problems for the scalar systems, and then extends the results to the vector systems. The results of numerical simulation are presented in Section 4. Conclusions are stated in Section 5.

2. Problem Formulation

We consider in this paper stabilization of discrete-time linear time-invariant (LTI) systems where the sensors and controller are geographically separated and connected via a bandwidth-limited and stochastic dropout digital communication channel. The plant is described by the state equation

$$X(k+1) = \mathbf{A}X(k) + \mathbf{B}U(k) + \mathbf{F}W(k) \qquad (1)$$

where $X(k) \in \mathbb{R}^n$ is the measurable state, $U(k) \in \mathbb{R}^l$ is the control input, and $W(k) \in \mathbb{R}^g$ is the disturbance. $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{F}$ are known constant matrices with appropriate dimensions. Here, $X(0)$ and $W(k)$, $\forall k \in \mathbb{N}$ are mutually independent random variables satisfying $E\|X(0)\|^2 < \phi_0 < \infty$ and $\sup_{k\in\mathbb{N}} E\|W(k)\|^2 < \phi_W < \infty$. Assume that the plant is unstable but the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable.

The information of plant states is transmitted via a noisy digital communication channel which is memoryless. Our model of the channel neglects channel propagation delays and focuses on the unreliability of the connection. As in [24] and [25], assume that the sensors and controller are connected via a lossy forward digital channel, and a reception/dropout acknowledgement is transmitted over an additional perfect (without packet losses and transmission errors) feedback channel. The packet dropout process in the forward channel is modeled as a time-homogenous Markov process $\{\gamma_k\}_{k\geq 0}$. Here, we set $\gamma_k = 1$ when the packet has been successfully delivered to the decoder, and set $\gamma_k = 0$ corresponding to the dropout of the packet. The Markov chain has a transition probability matrix defined by

$$(P(\gamma_{k+1} = j|\gamma_k = i))_{i,j\in s} = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix} \qquad (2)$$

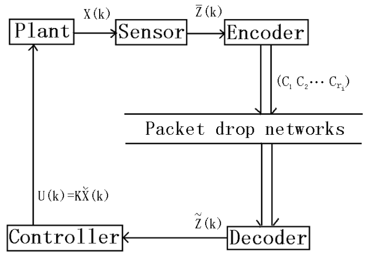

Figure 1. Networked control systems

where $s = \{0, 1\}$ is the state space of the Markov chain. Notice that the failure rate $p = P(\gamma_{k+1} = 0|\gamma_k = 1)$ and the recovery rate $q = P(\gamma_{k+1} = 1|\gamma_k = 0)$. Let $0 < p, q < 1$ such that the Markov chain is ergodic (see [25]).

Then, we may set $\gamma_0 = 1$ due to the ergodic property of the Markov chain. Let $\{T_d\}_{d\geq 0}$ denote the stochastic time sequence and let $T_0 = 1$. Then we define

$$T_d = \inf\{k : k \geq T_{d-1}, \gamma_k = 1\} + 1.$$

Let $\bar{T}_d := T_d - T_{d-1}$ denote the time duration between two successive packet reception times. As in [25], $\bar{T}_d$ is an independent and identically distributive (IID) random variable with the distribution expressed as

$$P(\bar{T}_d = i) = \begin{cases} 1-p, & \text{when } i = 1 \\ pq(1-q)^{i-2}, & \text{when } i > 1. \end{cases} \qquad (3)$$

If system matrix $\mathbf{A}$ has only real eigenvalues each with geometric multiplicity one, let $\mathbf{H}$ be the unitary matrix that diagonalizes $\mathbf{A} = \mathbf{H}'\Lambda\mathbf{H}$ where $\Lambda = \text{diag}[\lambda_1, \cdots, \lambda_n]$ and λ_i denotes the ith eigenvalue of $\mathbf{A}$ $(i = 1, \cdots, n)$. Otherwise, for general matrix $\mathbf{A}$, we have $\Lambda = \text{diag}[J_1, \cdots, J_m]$ where each $J_i (i = 1, \cdots, m)$ is a Jordan block of dimension (geometric multiplicity) n_i. Clearly $n_1 + \cdots + n_m = n$. It is derived in [9] that there are the same results for the two cases above. Thus, we can restrict our attention to the first case. Considering this case avoids extraneous complexity and makes our conclusions most transparent.

The coding technique presented in this paper is an adaptive differential coding strategy which was introduced in [26]. Let $\hat{X}(k)$ denote the decoder's estimate of $X(k)$ on the basis of the channel output. Difine

$$\bar{X}(k) = \mathbf{H}X(k),$$
$$\tilde{X}(k) = \mathbf{H}\hat{X}(k).$$

Then, the system (1) may be rewritten as

$$\bar{X}(k+1) = \Lambda\bar{X}(k) + \mathbf{HB}U(k) + \mathbf{HF}W(k).$$

A control law of the form

$$U(k) = \mathbf{K}\breve{X}(k) \tag{4}$$

is implemented where

$$\breve{X}(k) := \begin{cases} \hat{X}(k), & \text{if } \gamma_k = 1 \\ \grave{X}(k), & \text{if } \gamma_k = 0. \end{cases}$$

Here we define

$$\grave{X}(k) := (\mathbf{A} + \mathbf{BK})\breve{X}(k-1),$$
$$\vec{X}(k) := \mathbf{H}\grave{X}(k).$$

Let $\breve{X}(0) = \hat{X}(0)$ (since $\gamma_0 = 1$). Here, $\grave{X}(k)$ denotes the centroid of the uncertain region of $X(k)$ and $\vec{X}(k)$ denotes the centroid of the uncertain region of $\bar{X}(k)$. Define

$$Z(k) := X(k) - \grave{X}(k),$$
$$\bar{Z}(k) := \bar{X}(k) - \vec{X}(k).$$

Clearly, $\bar{Z}(k) = \mathbf{H}Z(k)$. Then, $\bar{Z}(k)$ will be quantized, coded and transmitted via a noisy digital channel. Let $\tilde{Z}(k)$ denote the quantization value of $\bar{Z}(k)$ and let $\hat{Z}(k) := \mathbf{H}'\tilde{Z}(k)$. Then we may define

$$\bar{V}(k) := \bar{Z}(k) - \tilde{Z}(k),$$
$$V(k) := \mathbf{H}'\bar{V}(k)$$

where $\bar{V}(k)$ denotes the quantization error with zero mean.

Let $\bar{Z}(k) := [\bar{z}_1(k)\ \bar{z}_2(k) \cdots \bar{z}_n(k)]'$. Similar to that in [27], the quantization scheme is presented. Given a positive integer M_i and a nonnegative real number $\Delta_i(k)$ $(i = 1, \cdots, n)$, define the quantizer $q : \mathbb{R} \to \mathbb{Z}$ with sensitivity $\Delta_i(k)$ and saturation value M_i by the formula

$$q(\bar{z}_i(k)) = \begin{cases} M^+, & \text{if } \bar{z}_i(k) > (M_i + 1/2)\Delta_i(t) \\ M^-, & \text{if } \bar{z}_i(k) \le -(M_i + 1/2)\Delta_i(t) \\ \lfloor \frac{\bar{z}_i(k)}{\Delta_i(t)} + \frac{1}{2} \rfloor, & \text{if } \bar{z}_i(k) > -(M_i + 1/2)\Delta_i(t) \\ & \text{and } \bar{z}_i(k) \le (M_i + 1/2)\Delta_i(t) \end{cases} \tag{5}$$

where $\lfloor \bar{z} \rfloor := \max\{k \in \mathbb{Z} := k < \bar{z}, \bar{z} \in \mathbb{R}\}$. The indexes M^+ and M^- will be employed if the quantizer saturates. The scheme to be used here is based on the hypothesis that it is possible to change the sensitivity (but not the saturation value) of the quantizer on the basis of available quantized measurements. The quantizer may counteract disturbances by switching repeatedly between “zooming out" and “zooming in" (see [27]).

Based on the quantization scheme given above, we can construct a code with codeword length r_i $(i = 1, \cdots, n)$. Let $(c_1 c_2 \cdots c_{r_i})$ denote the codeword corresponding to $\bar{z}_i(k)$. Namely, $\bar{z}_i(k)$ is quantized, encoded, and transformed into the r_i bits for transmission. Then we may compute $c_j \in \{0, 1\}$ $(j = 1, \cdots, r_{i-1})$

$$(c_1 c_2 \cdots c_{r_i - 1}) = \arg\max_{(c_1 c_2 \cdots c_{r_i - 1})} \textstyle\sum_{j=1}^{r_i - 1} c_j 2^{j-1} \tag{6}$$

subject to

$$\textstyle\sum_{j=1}^{r_i - 1} c_j 2^{j-1} \le |\lfloor \frac{\bar{z}_i(k)}{\Delta_i(k)} + \frac{1}{2} \rfloor|.$$

Furthermore, we set $c_{r_i} = 0$ when $\bar{z}_i(k) \ge 0$ and set $c_{r_i} = 1$ when $\bar{z}_i(k) < 0$. This implies that

$$r_i = \log_2 M_i + 1.$$

Then the data rate of the channel is given by

$$R = \textstyle\sum_{i=1}^{n} r_i \text{ (bits/sample)}.$$

Since the packet acknowledgement is transmitted over a perfect feedback channel, the encoder and decoder have across to $\breve{X}(k)$ such that they may update their estimator and scaling in the same manner. The value of $\hat{Z}(k)$ may be computed on the basis of the channel output at the decoder. Notice that

$$\hat{X}(k) = \grave{X}(k) + \hat{Z}(k).$$

Thus, $\hat{X}(k)$ may be obtained at the decoder when $\gamma_k = 1$.

As in [21], [25], etc, the system (1) is said to be mean square stabilization if for any initial state $X(0)$, there exists a control policy such that the states of the closed-loop system are uniformly bounded in the mean square sense

$$\limsup_{k \to \infty} E\|X(k)\|^2 < \infty. \tag{7}$$

The main problem is to present conditions on the data rate of the channel in relation to the transition probability matrix such that there exists a quantization, coding and control scheme to stabilize the system (1) in the mean sense (7) over a noisy digital channel with limited data rates.

3. Quantized Feedback Control Over Packet Dropout Channels

This section first deals with stabilization problems for the scalar system of the form

$$x(k+1) = \lambda x(k) + bu(k) + w(k) \tag{8}$$

where $|\lambda| \geq 1$ and $b \neq 0$. Here $w(k)$ is the disturbance with $E[w^2(k)] = \phi_w$. Considering this case avoids extraneous complexity, and makes our conclusions most transparent. Next, we extend the results to the vector systems.

In this case, let $\hat{x}(k)$ denote the decoder's estimate of $x(k)$. A control law of the form

$$u(k) = K\breve{x}(k) \tag{9}$$

is implemented where

$$\breve{x}(k) := \begin{cases} \hat{x}(k), & \text{if } \gamma_k = 1 \\ \grave{x}(k), & \text{if } \gamma_k = 0 \end{cases}$$

where $\grave{x}(k) := (\lambda + bK)\breve{x}(k-1)$. Then, let $z(k) := x(k) - \grave{x}(k)$ be quantized, encoded and transmitted over noisy digital channels. Let $\hat{z}(k)$ denote the quantization value of $z(k)$, and let $v(k)$ denote the quantization error with zero mean. Namely,

$$\hat{z}(k) = \Delta(k)q(z(k)),$$
$$z(k) = \hat{z}(k) + v(k).$$

This implies that

$$\hat{x}(k) = \grave{x}(k) + \hat{z}(k).$$

It means that the estimate of $x(k)$ is obtained on the basis of the channel output at the decoder.

Now, we present a sufficient condition for mean square stabilization of the system (8). The main task here is to discuss the effect of the disturbance on stabilization of the unstable plant over a noisy, bandwidth-limited digital channel. The conclusion is summarized below:

Theorem 3.1: Consider the system (8). Assume that the packet dropout process of the forward channel is a time-homogeneous Markov process with the transition probability matrix (2). A control law of the form (9) is implemented subject to $|\lambda + bK| < 1$. Then, there exists a quantization scheme of the form (5), a coding scheme of the form (6), and a control scheme of the form (9) to stabilize the system (8) in the mean square sense (7) if the following conditions hold:

- The probability q of the channel recovering from packet dropping satisfies the following inequality:
$$q > 1 - \tfrac{1}{\lambda^2};$$
- The data rate R satisfies the following inequality:
$$R > \max\{(1 + \tfrac{1}{q})\log_2|\lambda|,\ \log_2|\lambda| + \tfrac{1}{2}\log_2[(1 + \tfrac{p(\lambda^2-1)}{1-(1-q)\lambda^2}]\} \text{ (bits/sample)}.$$

Proof: Consider the closed-loop scalar system

$$x(k+1) = \lambda x(k) + bK\breve{x}(k) + w(k)$$

which we can also write as

$$x(k+1) = \lambda(x(k) - \breve{x}(k)) + (\lambda + bK)\breve{x}(k) + w(k).$$

Notice that

$$\grave{x}(k+1) = (\lambda + bK)\breve{x}(k),$$
$$x(k+1) = \grave{x}(k+1) + z(k+1).$$

Then,

$$z(k+1) = \begin{cases} \lambda v(k) + w(k), & \text{when } \gamma(k) = 1 \\ \lambda z(k) + w(k), & \text{when } \gamma(k) = 0. \end{cases} \tag{10}$$

This implies

$$z(T_d) = \lambda v(T_d - 1) + w(T_d - 1)$$

and

$$z(T_d - 1) = \begin{cases} \Sigma_{i=0}^{\bar{T}_d-2}\lambda^{\bar{T}_d-2-i}w(T_{d-1}+i) + \lambda^{\bar{T}_d-1}z(T_{d-1}), & \text{when } \bar{T}_d > 1 \\ z(T_{d-1}), & \text{when } \bar{T}_d = 1. \end{cases}$$

It follows in each duration $\bar{T}_d$ (i.e., $k \in [T_{d-1}, T_d]$) that

$$E[z^2(T_d)] = \lambda^2 E[v^2(T_d - 1)] + \phi_w \tag{11}$$

and

$$E[z^2(T_d - 1)] = \begin{cases} \lambda^{2(\bar{T}_d-1)}E[z^2(T_{d-1})] + \Sigma_{i=0}^{\bar{T}_d-2}\lambda^{2(\bar{T}_d-2-i)}\phi_w, & \text{when } \bar{T}_d > 1 \\ E[z^2(T_{d-1})], & \text{when } \bar{T}_d = 1. \end{cases} \tag{12}$$

It follows from the quantization scheme (5) that

$$R = \frac{1}{2}\log_2\frac{E[z^2(T_d - 1)]}{E[v^2(T_d - 1)]}.$$

It means that

$$E[z^2(T_d - 1)] = 2^{2R}E[v^2(T_{d-1})]. \tag{13}$$

By summing (11), (12) and (13), we obtain that

$$E[z^2(T_d)] = \frac{\lambda^{2\bar{T}_d}}{2^{2R}}E[z^2(T_{d-1})] + \Phi(\bar{T}_d) \tag{14}$$

where

$$\Phi(\bar{T}_d) := \begin{cases} \frac{\lambda^2}{2^{2R}}\Sigma_{i=0}^{\bar{T}_d-2}\lambda^{2(\bar{T}_d-2-i)}\phi_w + \phi_w, & \text{when } \bar{T}_d > 1 \\ \phi_w, & \text{when } \bar{T}_d = 1. \end{cases}$$

It follows immediately that

$$E[z^2(T_d)] = \frac{\lambda^{2(\bar{T}_1+\bar{T}_2+\cdots+\bar{T}_d)}}{2^{2nR}} E[z^2(T_0)] + \bar{\Phi}(d)$$

where

$$\begin{aligned}\bar{\Phi}(d) := \Phi(\bar{T}_d) + \tfrac{\lambda^{2\bar{T}_d}}{2^{2R}}\Phi(\bar{T}_{d-1}) + \tfrac{\lambda^{2(\bar{T}_d+\bar{T}_{d-1})}}{2^{4R}}\Phi(\bar{T}_{d-2}) \\ + \cdots + \tfrac{\lambda^{2(\bar{T}_d+\bar{T}_{d-1}+\cdots+\bar{T}_2)}}{2^{2(d-1)R}}\Phi(\bar{T}_1).\end{aligned}$$

By the strong law of large numbers, we see that

$$\frac{1}{d}\Sigma_{i=1}^{d}\bar{T}_i = E\bar{T}_d = 1 + \frac{1}{q}.$$

When d is large enough, it follows that

$$E[z^2(T_d)] = (\tfrac{\lambda^{1+\frac{1}{q}}}{2^R})^{2d} E[z^2(T_0)] + \bar{\Phi}(d). \tag{15}$$

Notice that $\bar{\Phi}(d)$ is still a function of $\bar{T}_1$, $\bar{T}_2$, $\cdots$, and $\bar{T}_d$. Then, we may compute the expectation of $\bar{\Phi}(d)$ over $\bar{T}_1$, $\bar{T}_2$, $\cdots$ and $\bar{T}_d$. Here, we define

$$\begin{aligned}\Phi_1 &:= E\Phi(\bar{T}_1) = E\Phi(\bar{T}_2) = \cdots = E\Phi(\bar{T}_d) \\ &= \begin{cases} (\frac{\lambda^2}{2^{2R}}\frac{p}{1-\lambda^2(1-q)} + 1)\phi_w, & \text{when } q > 1 - \frac{1}{\lambda^2} \\ \infty, & \text{when } q \le 1 - \frac{1}{\lambda^2}\end{cases}\end{aligned}$$

and

$$\begin{aligned}\Phi_2 &:= E\tfrac{\lambda^{2\bar{T}_1}}{2^{2R}} = E\tfrac{\lambda^{2\bar{T}_2}}{2^{2R}} = \cdots = E\tfrac{\lambda^{2\bar{T}_d}}{2^{2R}} \\ &= \begin{cases} \frac{\lambda^2}{2^{2R}}(\frac{p(\lambda^2-1)}{1-\lambda^2(1-q)} + 1), & \text{when } q > 1 - \frac{1}{\lambda^2} \\ \infty, & \text{when } q \le 1 - \frac{1}{\lambda^2}\end{cases}\end{aligned}$$

Thus, it follows that

$$\begin{aligned}E\bar{\Phi}(d) &= \Phi_1 + \Phi_2\Phi_1 + \Phi_2^2\Phi_1 + \cdots + \Phi_2^{d-1}\Phi_1 \\ &= \tfrac{1-\Phi_2^d}{1-\Phi_2}\Phi_1.\end{aligned}$$

If the data rate R satisfies the following condition:

$$R > \log_2|\lambda| + \tfrac{1}{2}\log_2[(1 + \tfrac{p(\lambda^2-1)}{1-(1-q)\lambda^2}] \text{ (bits/sample)}, \tag{16}$$

then, $\Phi_2 < 1$ holds. This implies that

$$\lim_{d\to\infty} E\bar{\Phi}(d) = \frac{\Phi_1}{1-\Phi_2} < \infty.$$

Furthermore, if the data rate R satisfies the following condition:

$$R > (1 + \tfrac{1}{q})\log_2|\lambda| \text{ (bits/sample)}, \tag{17}$$

then, $\frac{\lambda^{1+\frac{1}{q}}}{2^R} < 1$ holds. By summing (15), (16), and (17), we see that, if the data rate R satisfies the following condition:

$$\begin{aligned}R > \max\{&(1 + \tfrac{1}{q})\log_2|\lambda|,\ \log_2|\lambda| \\ &+ \tfrac{1}{2}\log_2[(1 + \tfrac{p(\lambda^2-1)}{1-(1-q)\lambda^2}]\} \text{ (bits/sample)}\end{aligned}$$

and $q > 1 - \frac{1}{\lambda^2}$ holds, then

$$\lim_{k\to\infty} E[z^2(k)] = \lim_{d\to\infty} E[z^2(T_d)] = \frac{\Phi_1}{1-\Phi_2} < \infty.$$

Furthermore, notice that

$$\begin{aligned}x(k) &= \check{x}(k) + z(k), \\ x(k+1) &= \check{x}(k+1) + z(k+1).\end{aligned}$$

Then, we may rewrite the system (8) as

$$\check{x}(k+1) = \lambda\check{x}(k) + bK\check{x}(k) + \lambda z(k) - z(k+1) + w(k).$$

Substitute (10) into the equality above, and obtain

$$\check{x}(k+1) = \begin{cases}(\lambda + bK)\check{x}(k), & \text{when } \gamma_k = 0 \\ (\lambda + bK)\check{x}(k) + (\lambda + bK)\hat{z}(k), & \text{when } \gamma_k = 1.\end{cases}$$

This implies

$$\begin{aligned}\check{x}(T_d) &= (\lambda + bK)\check{x}(T_d - 1) + (\lambda + bK)\hat{z}(T_d - 1), \\ \check{x}(T_d - 1) &= (\lambda + bK)^{\bar{T}_d - 1}\check{x}(T_{d-1}).\end{aligned}$$

Thus,

$$\check{x}(T_d) = (\lambda + bK)^{\bar{T}_d}\check{x}(T_{d-1}) + (\lambda + bK)\hat{z}(T_d - 1).$$

Notice that

$$|\lambda + bK| < 1$$

and

$$\lim_{k\to\infty} E[\hat{z}^2(k)] = \lim_{d\to\infty} E[\hat{z}^2(T_d - 1)] < \lim_{k\to\infty} E[z^2(k)] < \infty.$$

Clearly, it holds that

$$\lim_{k\to\infty} E[\check{x}^2(k)] = \lim_{d\to\infty} E[\check{x}^2(T_d)] < \infty.$$

Thus, it follows that

$$\lim_{k\to\infty} E\|x(k)\|^2 = \lim_{k\to\infty} E[\check{x}^2(k)] + \lim_{k\to\infty} E[z^2(k)] < \infty.$$

□

Remark 3.1:

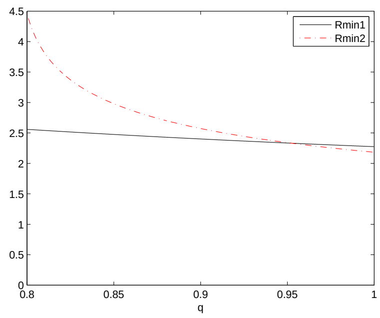

Figure 2. Curves of R_{min1} and R_{min2} when $1 > q > 1 - \frac{1}{|\lambda|^2}$

- It is shown in Theorem 3.1 that not only the data rate R but also the probability q of the channel recovering from packet dropping is needed to be large enough to guarantee mean square stabilization of the unstable plant.
- In comparison with [25], the condition in Theorem 3.1 is similar to that of [25]. Namely, the conditions:

$$q > 1 - \frac{1}{|\lambda|^2}$$

and

$$R > \log_2 |\lambda| + \frac{1}{2}\log_2[1 + \frac{p(|\lambda|^2-1)}{1-(1-q)|\lambda|^2}] \text{ (bits/sample)}$$

are satisfied to guarantee mean square stabilization of the system (8). Our result differs from [25] in that the data rate R for mean square stabilization satisfies an additional condition:

$$R > (1 + \frac{1}{q})\log_2 |\lambda| \text{ (bits/sample)}.$$

Namely, in Theorem 3.1, the data rate satisfies the condition:

$$R > \max\{(1 + \frac{1}{q})\log_2 |\lambda|,\ \log_2 |\lambda| + \frac{1}{2}\log_2[1 + \frac{p(|\lambda|^2-1)}{1-(1-q)|\lambda|^2}]\} \text{ (bits/sample)}$$

where we define

$$R_{min1} := (1 + \frac{1}{q})\log_2 |\lambda|,$$
$$R_{min2} := \log_2 |\lambda| + \frac{1}{2}\log_2[1 + \frac{p(|\lambda|^2-1)}{1-(1-q)|\lambda|^2}].$$

For example, let $\lambda = 2.2$ and $p = 0.85$. Then, the curves of R_{min1} and R_{min2} corresponding to q are given by Fig.2. It means that for some special cases, the condition $R > R_{min2}$ may not be sufficient to guarantee mean square stabilization of the system.

Now, we deal with quantized feedback control problems for the vector system, and extend the results above to the more general case. Then, we present the following result:

Theorem 3.2: Consider the system (1). Assume that the packet dropout process of the forward channel is a time-homogeneous Markov process with the transition probability matrix (2). A control law of the form (4) is implemented subject to the condition that all eigenvalues of $\mathbf{A} + \mathbf{BK}$ lie inside the unit circle. Define $\Xi := \{i : |\lambda_i| > 1\}$. Then, there exists a quantization scheme of the form (5), a coding scheme of the form (6), and a control scheme of the form (4) to stabilize the system (1) in the mean square sense (7) if the following conditions hold:

- The probability q of the channel recovering from packet dropping satisfies the following inequality:

$$q > \max_{i\in\Xi}\{1 - \frac{1}{\lambda_i^2}\};$$

- The data rate R satisfies the following inequality:

$$R > \max\{(1 + \frac{1}{q})\sum_{i\in\Xi}\log_2 |\lambda_i|,\ \sum_{i\in\Xi}[\log_2 |\lambda_i| + \frac{1}{2}\log_2(1 + \frac{p(|\lambda_i|^2-1)}{1-(1-q)|\lambda_i|^2})]\} \text{ (bits/sample)}.$$

Proof: Consider the closed-loop system

$$X(k+1) = \mathbf{A}X(k) + \mathbf{BK}\check{X}(k) + \mathbf{F}W(k)$$

which we can also write as

$$X(k+1) = \mathbf{A}(X(k) - \check{X}(k)) + (\mathbf{A} + \mathbf{BK})\check{X}(k) + \mathbf{F}W(k).$$

Notice that

$$\grave{X}(k+1) = (\mathbf{A} + \mathbf{BK})\check{X}(k),$$
$$X(k+1) = \grave{X}(k+1) + Z(k+1).$$

Then,

$$Z(k+1) = \begin{cases} \mathbf{A}V(k) + \mathbf{F}W(k), & \text{when } \gamma_k = 1 \\ \mathbf{A}Z(k) + \mathbf{F}W(k), & \text{when } \gamma_k = 0 \end{cases} \quad (18)$$

which is equivalent to

$$\bar{Z}(k+1) = \begin{cases} \Lambda\bar{V}(k) + \mathbf{HF}W(k), & \text{when } \gamma_k = 1 \\ \Lambda\bar{Z}(k) + \mathbf{HF}W(k), & \text{when } \gamma_k = 0. \end{cases} \quad (19)$$

This implies that

$$\bar{Z}(T_d) = \Lambda\bar{V}(T_d-1) + \mathbf{HF}W(T_d-1),$$

$$\bar{Z}(T_d-1) = \begin{cases} \Sigma_{j=0}^{\bar{T}_d-2}\Lambda^{\bar{T}_d-j-2}\mathbf{HF}W(T_{d-1}+j) \\ \qquad +\Lambda^{\bar{T}_d-1}\bar{Z}(T_{d-1}), \text{ when } \bar{T}_d > 1 \\ \bar{Z}(T_{d-1}), \qquad \text{when } \bar{T}_d = 1. \end{cases}$$

Here, we define $\Sigma_W := EW(k)W(k)'$ and define $\bar{\phi}_i := (\mathbf{HF}\Sigma_W\mathbf{F}'\mathbf{H}')_{ii}$ where $(\cdot)_{ij}$ denotes an entry of a matrix $(i, j = 1, \cdots, n)$. In each duration $\bar{T}_d$(i.e., $k \in [T_{d-1}, T_d]$), we have

$$\begin{aligned} E\|\bar{Z}(T_d)\|^2 &= \text{tr}[\Sigma_{\bar{Z}(T_d)}] \\ &= \text{tr}[\Lambda^2\Sigma_{\bar{V}(T_d-1)}] + \text{tr}[\mathbf{HF}\Sigma_W\mathbf{F}'\mathbf{H}'] \end{aligned}$$

and

$$\text{tr}[\Sigma_{\bar{Z}(T_d-1)}] = \begin{cases} \Sigma_{j=0}^{\bar{T}_d-2}\text{tr}[\Lambda^{2(\bar{T}_d-2-j)}\mathbf{HF}\Sigma_W\mathbf{F}'\mathbf{H}'] \\ \quad +\text{tr}[\Lambda^{2(\bar{T}_d-1)}\Sigma_{\bar{Z}(T_{d-1})}], \text{ when } \bar{T}_d > 1 \\ \text{tr}[\Sigma_{\bar{Z}(T_{d-1})}], \qquad \text{when } \bar{T}_d = 1 \end{cases}$$

which are equivalent to

$$E[\bar{z}_i^2(T_d)] = \lambda_i^2 E[\bar{v}_i^2(T_d-1)] + \bar{\phi}_i \tag{20}$$

and

$$E[\bar{z}_i^2(T_d-1)] = \begin{cases} \lambda_i^{2(\bar{T}_d-1)}E[\bar{z}_i^2(T_{d-1})] \\ +\Sigma_{j=0}^{\bar{T}_d-2}\lambda_i^{2(\bar{T}_d-2-j)}\bar{\phi}_i, \text{ when } \bar{T}_d > 1 \\ E[\bar{z}_i^2(T_{d-1})], \qquad \text{when } \bar{T}_d = 1. \end{cases} \tag{21}$$

Here we quantize, encode each $\bar{z}_i(T_d-1)$, and transmit the information of each $\bar{z}_i(T_d-1)$ by the data rate r_i (bits/sample). Then, it follows from the quantization (5) that the data rate r_i corresponding to $\bar{z}_i(T_d-1)$ is given by

$$r_i = \frac{1}{2}\log_2\frac{E[\bar{z}_i^2(T_d-1)]}{E[\bar{v}_i^2(T_d-1)]} \text{ (bits/sample)}.$$

Namely,

$$E[\bar{z}_i^2(T_d-1)] = 2^{2r_i}E[\bar{v}_i^2(T_d-1)]. \tag{22}$$

By summing (20), (21), and (22), we obtain that

$$E[z_i^2(T_d)] = \frac{\lambda_i^{2\bar{T}_d}}{2^{2r_i}}E[z_i^2(T_{d-1})] + \Psi(\bar{T}_d)$$

where

$$\Psi(\bar{T}_d) := \begin{cases} \frac{\lambda_i^2}{2^{2r_i}}\Sigma_{j=0}^{\bar{T}_d-2}\lambda_i^{2(\bar{T}_d-2-j)}\bar{\phi}_i + \bar{\phi}_i, & \text{when } \bar{T}_d > 1 \\ \bar{\phi}_i, & \text{when } \bar{T}_d = 1. \end{cases}$$

The equality above is similar to (14). Using the same techniques as in the proof of Theorem 3.1, we can show that, there exists a quantization scheme of the form (5), a coding scheme of the form (6), and a control scheme of the form (4) such that

$$\lim_{k\to\infty} E\|Z(k)\|^2 < \infty \tag{23}$$

holds if the data rate R satisfies the following condition:

$$\begin{aligned} R > \max\{&(1+\tfrac{1}{q})\textstyle\sum_{i\in\Xi}\log_2|\lambda_i|, \ \sum_{i\in\Xi}[\log_2|\lambda_i| \\ &+\tfrac{1}{2}\log_2(1+\tfrac{p(|\lambda_i|^2-1)}{1-(1-q)|\lambda_i|^2})]\} \text{ (bits/sample)} \end{aligned}$$

and $q > \max_{i\in\Xi}\{1 - \frac{1}{\lambda_i^2}\}$ holds where $\Xi := \{i : |\lambda_i| > 1\}$.

Furthermore, notice that

$$X(k) = \check{X}(k) + Z(k).$$

Then, we can also rewrite the system (1) as

$$\check{X}(k+1) = \mathbf{A}\check{X}(k) + \mathbf{BK}\check{X}(k) + \mathbf{A}Z(k) - Z(k+1) + \mathbf{F}W(k).$$

Substitute (18) into the equality above and obtain

$$\check{X}(k+1) = \begin{cases} (\mathbf{A}+\mathbf{BK})\check{X}(k), & \text{when } \gamma_k = 0 \\ (\mathbf{A}+\mathbf{BK})\check{X}(k) + (\mathbf{A}+\mathbf{BK})\hat{Z}(k), & \text{when } \gamma_k = 1. \end{cases}$$

This implies that

$$\begin{aligned} &\check{X}(T_d) = (\mathbf{A}+\mathbf{BK})\check{X}(T_d-1) + (\mathbf{A}+\mathbf{BK})\hat{Z}(T_d-1), \\ &\check{X}(T_d-1) = (\mathbf{A}+\mathbf{BK})^{\bar{T}_d-1}\check{X}(T_{d-1}). \end{aligned}$$

Thus, it holds that

$$\check{X}(T_d) = (\mathbf{A}+\mathbf{BK})^{\bar{T}_d}\check{X}(T_{d-1}) + (\mathbf{A}+\mathbf{BK})\hat{Z}(T_d-1).$$

Since

$$\lim_{d\to\infty} E\|\hat{Z}(T_d-1)\|^2 < \lim_{k\to\infty} E\|Z(k)\|^2 < \infty$$

holds and all eigenvalues of $\mathbf{A}+\mathbf{BK}$ lie inside the unit circle, it holds that

$$\lim_{k\to\infty} E\|\check{X}(k)\|^2 = \lim_{d\to\infty} E\|\check{X}(T_d)\|^2 < \infty.$$

Thus, it follows that

$$\lim_{k\to\infty} E\|X(k)\|^2 = \lim_{k\to\infty} E\|\check{X}(k)\|^2 + \lim_{k\to\infty} E\|Z(k)\|^2 < \infty.$$

□

Remark 3.2:

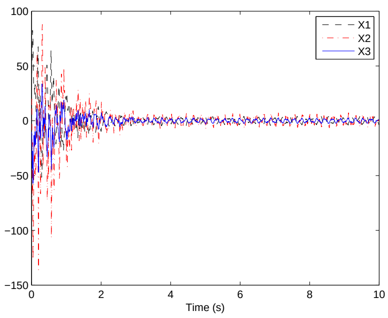

Figure 3. The system state responses with the disturbances when $q = 0.85$

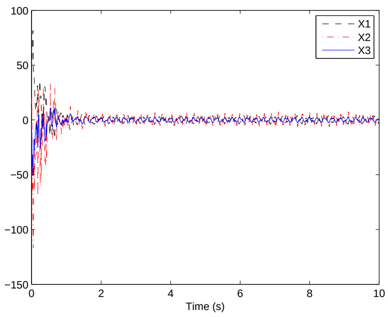

Figure 4. The system state responses with the disturbances when $q = 0.95$

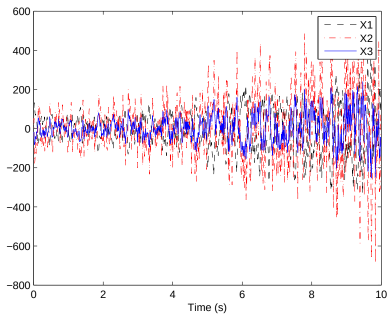

Figure 5. The system state responses with the disturbances when $q = 0.45$

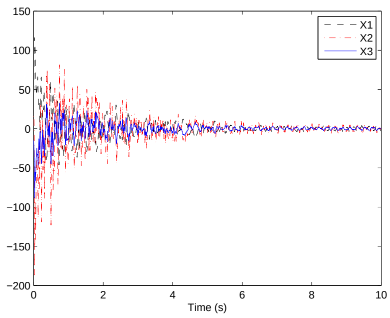

Figure 6. The system state responses with the different initial condition $X(0)$ when $q = 0.85$

- Notice that [25] considered noise free vector systems, but did not consider vector systems with the disturbances. However, the case with the disturbances is a more general and interesting case, which also faces some difficult challenges (such as how to allocate bits to each unstable state variable in this case).
- It is shown in Theorem 3.2 that the stable part does not play any role on the condition on the data rate for stabilization, which is similar to those of [11], [20], [21], etc.

4. Numerical Example

We consider a class of networked control problems which arises in the coordinated motion control of autonomous and semiautonomous mobile agents, e.g., unmanned air vehicles (UAVs), unmanned ground vehicles (UGVs), and unmanned underwater vehicles (UUVs). Here, we present a numerical example to illustrate the effectiveness of the proposed adaptive differential coding strategy and the predictive control law. Three of the states of an unmanned air vehicle evolve in discrete-time according to

$$X(k+1) = \begin{bmatrix} 2.2312 & 1.5423 & 1.2351 \\ 0.2142 & 2.3225 & 1.2112 \\ 0.7815 & 0.2526 & 3.1721 \end{bmatrix} X(k) + \begin{bmatrix} 2.1223 \\ 3.4411 \\ 1.5335 \end{bmatrix} u(k) + W(k).$$

The feedback gain is given by $\mathbf{K} = [1.0485 \;\; 0.1339 \;\; 2.3538]$. Let $X(0) = [50 \;\; 30 \;\; -50]'$. The proposed predictive control law (4) with the quantization scheme (5) subject to the minimum data rate limitation on the basis of the condition in Theorem 3.2 is employed.

Let $p = 0.85$, $R = 240$ (bits/s), and $\phi_W = 1$. If we set $q = 0.85$, we may obtain the corresponding simulation given in Fig.3. It is shown that the system with disturbances may be stabilizable in the mean square sense (7) if both the data rate and the probability q are large enough. If we set $q = 0.95$, we may obtain the corresponding simulation given in Fig.4. It states the larger the probability q of the channel recovering from packet dropping, the better the control performance obtained. On the contrary, if we set $q = 0.45$, we may obtain the corresponding simulation given in Fig.5. It is shown that the system is not stabilizable if the probability q is smaller than $\max\{1 - \frac{1}{\lambda_i^2}\}$.

We may also set $X(0) = [100\ -100\ -50]'$, $p = 0.85$, $R = 240$ (bits/s), and $\phi_W = 1$. Here, we consider the case with the different initial condition $X(0)$, and obtain the corresponding simulation given in Fig.6. It states that the initial condition has no effect on stabilization of the system.

5. Conclusion

In this paper, we considered quantized feedback control problems, and presented sufficient conditions for mean square stabilization of linear time-invariant systems over noisy, bandwidth-limited digital communication channels. An adaptive differential coding strategy was employed, which benefits from many advantages, such as minimum bit rate transmission, good stability performance, etc. It was shown that our results are less conservative than those of the literature. The simulation results have illustrated the effectiveness of the quantization, coding and control scheme.

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