EXTREMES OF A BIVARIATE PARETO DISTRIBUTION

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Abstract. Distributional properties of \( \min(X, Y) \) and \( \max(X, Y) \) are studied when \((X, Y)\) has a bivariate Pareto distribution. Extensions are given to the multivariate case.

Keywords: Bivariate Pareto distributions, extremes, moments.

1. Introduction

Bivariate Pareto distributions are popular models in many applied areas. They are very versatile and a variety of uncertainties can be usefully modeled by them. We mention: modeling of radiation carcinogenesis (Rachev et al. [1]), performance measures for general systems (Nadarajah and Kotz [2]), reliability (Hanagal [3]; Navarro et al. [4]), modeling of drought (Nadarajah [5]), modeling of dependent heavy tailed risks with a non-zero probability of simultaneous loss (Asimit et al. [6]), and modeling of daily exchange rate data (Papadakis and Tsionas [7]).

Let \((X, Y)\) be a bivariate Pareto random vector. In the mentioned applications, \(X\) and \(Y\) could be the lifetimes of two components, the magnitudes of stress and strength components, drought intensities for two regions, risks for two insurance events, exchange rates in two time periods, and so on. So, it is important to know which of the two variables, \(X\) and \(Y\), is larger or smaller.

Let \(S = \min(X, Y)\) and \(T = \max(X, Y)\). The aim of this note is to study the distributions of \(S\) and \(T\) when \((X, Y)\) has a bivariate Pareto distribution. Studies of this kind have been considered by several authors. Ker [8] studies the distribution of \(T\) when \((X, Y)\) has a bivariate normal distribution. Lien [9] studies the distributions of \(S\) and \(T\) when \((X, Y)\) has a bivariate lognormal distribution. Aksamitis and Burauskaite-Harju [10] study the distribution of \(\max(X_1, X_2, \ldots, X_n)\) and its moments when \((X_1, X_2, \ldots, X_n)\) has a multivariate normal distribution.

It seems, however, that the distributions of \(S\) and \(T\) have not been studied when \((X, Y)\) has a bivariate Pareto distribution. This note provides the first such study. We take \((X, Y)\) to have the simplest bivariate Pareto distribution due to Muliere and Scarsini [11]: the one given by the following joint survival function:

\[
F_{X,Y}(x,y) = \left( \frac{x}{\beta} \right)^{-\lambda_1} \left( \frac{y}{\beta} \right)^{-\lambda_2} \times \left\{ \max \left( \frac{x}{\beta}, \frac{y}{\beta} \right) \right\}^{-\lambda_0}
\]  

(1.1)

for \(\lambda_i > 0, i = 0, 1, 2\), where \(0 < \beta \leq \min(x, y) < \infty\). The corresponding joint cumulative distribution
function is

$$F_{X,Y}(x, y) = 1 - \left( \frac{x}{\beta} \right)^{-\lambda_0 - \lambda_1} - \left( \frac{y}{\beta} \right)^{-\lambda_0 - \lambda_2} + \left( \frac{x}{\beta} \right)^{-\lambda_1} \left( \frac{y}{\beta} \right)^{-\lambda_2} \times \left\{ \max \left( \frac{x}{\beta}, \frac{y}{\beta} \right) \right\}^{-\lambda_0}.$$ 

The cumulative distribution functions of $S$ and $T$ are

$$F_S(s) = 1 - \left( \frac{s}{\beta} \right)^{-(\lambda_0 + \lambda_1 + \lambda_2)}$$

and

$$F_T(t) = 1 - \left( \frac{t}{\beta} \right)^{-\lambda_0 - \lambda_1} + \left( \frac{t}{\beta} \right)^{-\lambda_0 - \lambda_1 - \lambda_2} - \left( \frac{t}{\beta} \right)^{-\lambda_0 - \lambda_2}.$$ 

The corresponding probability density functions are:

$$f_S(s) = (\lambda_0 + \lambda_1 + \lambda_2) \beta^{\lambda_0 + \lambda_1 + \lambda_2} s^{-\lambda_0 - \lambda_1 - \lambda_2 - 1},$$

and

$$f_T(t) = (\lambda_0 + \lambda_1) \beta^{\lambda_0 + \lambda_1} \lambda_2 \lambda_0 - \lambda_1 - \lambda_2 - 1 - (\lambda_0 + \lambda_1 + \lambda_2) \beta^{\lambda_0 + \lambda_1 + \lambda_2} \times t^{-\lambda_0 - \lambda_1 - \lambda_2 - 1} + (\lambda_0 - \lambda_2) \beta^{\lambda_0 + \lambda_2} t^{-\lambda_0 - \lambda_2 - 1},$$

where $0 < \beta < t < s < \infty$ and $\lambda_i > 0$, $i = 0, 1, 2$.

In Section 2, we provide expressions for $E(S)$, $Var(S)$, $E(T)$, $Var(T)$, and examine the effects of $\lambda_i$ on them. Section 3 provides an extension to a multivariate case.

2. Main result

Theorem 1 is our main result.

**Theorem 1.** Let $(X, Y)$ be distributed according to (1.1). Let $S = \min(X, Y)$, $T = \max(X, Y)$ and $\lambda = \lambda_0 + \lambda_1 + \lambda_2$. Then

$$E(S) = \frac{\lambda \beta}{\lambda - 1}, \lambda > 1,$$

$$E(S^2) = \frac{\lambda \beta^2}{\lambda - 2}, \lambda > 2,$$

$$Var(S) = \frac{\beta^2 \lambda}{(\lambda - 2)(\lambda - 1)^2}, \lambda > 2,$$

$$E(T) = \frac{\beta (\lambda_0 + \lambda_1)}{\lambda_0 + \lambda_1 - 1} + \frac{\beta \lambda}{1 - \lambda} + \frac{\beta (\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 1},$$

$$E(T^2) = \frac{\beta^2 (\lambda_0 + \lambda_1)}{\lambda_0 + \lambda_1 - 2} + \frac{\beta^2 \lambda}{2 - \lambda} + \frac{\beta^2 (\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2},$$

$$Var(T) = \frac{\beta^2}{2 - \lambda} \left[ \frac{\lambda_0 + \lambda_1}{\lambda_0 + \lambda_1 - 2} + \frac{\lambda_0 + \lambda_2}{\lambda_0 + \lambda_2 - 2} \right] + \frac{\lambda}{1 - \lambda} \left( \frac{\lambda_0 + \lambda_1}{\lambda_0 + \lambda_1 - 1} + \frac{\lambda_0 + \lambda_2}{\lambda_0 + \lambda_2 - 1} \right)^2,$$

$$\lambda_0 > \max \{0, 2 - \lambda_1, 2 - \lambda_2\}.$$ 

Furthermore, we have the following statements holding:

(I) $E(S)$ is monotonically increasing with respect to $\beta$ for $\lambda > 1$,

(II) $E(S)$ is monotonically decreasing with respect to $\lambda$ and $\lambda_i$, $i = 0, 1, 2$ for $\lambda > 1$,

(III) $Var(S)$ is monotonically increasing with respect to $\beta$ for $\lambda > 2$,

(IV) $Var(S)$ is monotonically decreasing with respect to $\lambda$ and $\lambda_i$ for $i = 0, 1, 2$ for $\lambda > 2$.

(V) $E(T)$ is monotonically increasing with respect to $\beta$ for $\lambda_0 > \max \{1 - \lambda_1, 1 - \lambda_2, 0\}$,

(VI) $E(T)$ is monotonically decreasing with respect to $\lambda_i$ for $i = 0, 1, 2$ for $\lambda_0 > \max \{1 - \lambda_1, 1 - \lambda_2, 0\}$,

(VII) $Var(T)$ is monotonically increasing with respect to $\beta$,

(VIII) $Var(T)$ is monotonically decreasing with respect to $\lambda_i$ for $i = 0, 1, 2$. 

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**Proof:** The given expressions for $E(S)$, $E(S^2)$ and $\text{Var}(S)$ follow by using:

$$E(S) = \int_{\beta}^{\infty} \lambda \beta^3 s^{-\lambda} ds,$$

$$E(S^2) = \int_{\beta}^{\infty} \lambda \beta^4 s^{1-\lambda} ds,$$

and $\text{Var}(S) = E(S^2) - [E(S)]^2$. Since, using (1.2),

$$E(T) = \int_{\beta}^{\infty} t f_T(t) dt = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\beta}^{\infty} (\lambda_0 + \lambda_1) \beta^{\lambda_0 + \lambda_1} t^{-\lambda_0 - \lambda_1} dt$$

$$= \frac{\beta (\lambda_0 + \lambda_1)}{\lambda_0 + \lambda_1 - 1}, \lambda_0 + \lambda_1 > 1,$$

$$I_2 = \int_{\beta}^{\infty} -\lambda \beta^3 t^{-\lambda - 1} dt$$

$$= \frac{-\beta \lambda}{1 - \lambda}, \lambda > 1,$$

$$I_3 = \int_{\beta}^{\infty} (\lambda_0 + \lambda_2) \beta^{\lambda_0 + \lambda_2} t^{-\lambda_0 - \lambda_2} dt$$

$$= \frac{\beta (\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 1}, \lambda_0 + \lambda_2 > 1,$$

we obtain the given expression for $E(T)$. Similarly, using

$$E(T^2) = \int_{\beta}^{\infty} t^2 f_T(t) dt = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\beta}^{\infty} (\lambda_0 + \lambda_1) \beta^{\lambda_0 + \lambda_1} t^{-\lambda_0 - \lambda_1 + 1} dt$$

$$= \frac{\beta^2 (\lambda_0 + \lambda_1)}{\lambda_0 + \lambda_1 - 2}, \lambda_0 + \lambda_1 > 2,$$

$$I_2 = \int_{\beta}^{\infty} -\lambda \beta^3 t^{-\lambda + 1} dt$$

$$= \frac{-\beta \lambda}{2 - \lambda}, \lambda > 2,$$

$$I_3 = \int_{\beta}^{\infty} (\lambda_0 + \lambda_2) \beta^{\lambda_0 + \lambda_2} t^{-\lambda_0 - \lambda_2 + 1} dt$$

$$= \frac{\beta^2 (\lambda_0 + \lambda_2)}{\lambda_0 + \lambda_2 - 2}, \lambda_0 + \lambda_2 > 2,$$

and $\text{Var}(T) = E(T^2) - [E(T)]^2$, we obtain the given expression for $\text{Var}(T)$.

The remainder of the theorem is proved as follows:

(I) We have

$$\frac{\partial E(S)}{\partial \beta} = \frac{\lambda}{\lambda - 1} > 0, \lambda > 1,$$

so statement (I) follows.

(II) We have

$$\frac{\partial E(S)}{\partial \lambda_i} = -\frac{\beta}{(\lambda - 1)^2} < 0, \lambda > 1,$$

and

$$\frac{\partial E(S)}{\partial \lambda} = -\frac{\beta}{(\lambda - 1)^2} < 0, \lambda > 1,$$

so statement (II) follows.

(III) We have

$$\frac{\partial \text{Var}(S)}{\partial \beta} = \frac{2\beta \lambda}{(\lambda - 2)(\lambda - 1)^2}.$$<br>

Since $\beta > 0$ and $\lambda > 2$, $\partial \text{Var}(S)/\partial \beta > 0$, so statement (III) follows.

(IV) We have

$$\frac{\partial \text{Var}(S)}{\partial \lambda_i} = -2\beta^2 \frac{(\lambda^2 - 1 - \lambda)}{(\lambda - 1)^3 (\lambda - 2)^2}, i = 0, 1, 2,$$

and

$$\frac{\partial \text{Var}(S)}{\partial \lambda} = -2\beta^2 \frac{\lambda^2 - \lambda - 1}{(\lambda - 1)^4 (\lambda - 2)^2}.$$<br>

Since $\lambda > 2$ and $(\lambda^2 - \lambda - 1) > 0$, we have

$$\frac{\partial \text{Var}(S)}{\partial \lambda_i} < 0,$$

and

$$\frac{\partial \text{Var}(S)}{\partial \lambda} < 0,$$

so statement (IV) follows.

(V) We have

$$\frac{\partial E(T)}{\partial \beta} = \frac{\lambda_0 + \lambda_1}{\lambda_0 + \lambda_1 - 1} + \frac{\lambda_0 + \lambda_2}{\lambda_0 + \lambda_2 - 1} - \frac{\lambda}{\lambda - 1},$$

$$\lambda_0 > \max \{1 - \lambda_1, 1 - \lambda_2, 0\}.$$
Since \((\lambda_0 + \lambda_1)/(\lambda_0 + \lambda_1 - 1) > \lambda/\lambda - 1\),
\(\partial E(T)/\partial \beta > 0\), so statement (V) follows.

(VI) We have
\[
\frac{\partial E(T)}{\partial \lambda_0} = \beta \left[ \frac{1}{\lambda_0 + \lambda_1 - 1} + \frac{1}{\lambda_0 + \lambda_2 - 1} - \frac{1}{\lambda - 1} \right. \\
\left. - \frac{\lambda_0 + \lambda_1}{(\lambda_0 + \lambda_1 - 1)^2} + \frac{\lambda_0 + \lambda_2}{(\lambda_0 + \lambda_2 - 1)^2} - \frac{\lambda}{(\lambda - 1)^2} \right],
\]
and
\[
\frac{\partial E(T)}{\partial \lambda_1} = -\beta \lambda_2 \frac{\lambda_0 + 2\lambda_0 + 2\lambda_1 + 2\lambda_2}{(\lambda_0 + \lambda_1 - 1)^2 (\lambda - 1)^2},
\]
Since \(1/(\lambda_0 + \lambda_2 - 1) < (\lambda_0 + \lambda_1)/(\lambda_0 + \lambda_2 - 1)^2\) and
\(1/(\lambda_0 + \lambda_1 - 1) - 1/(\lambda - 1) < (\lambda_0 + \lambda_1)/(\lambda_0 + \lambda_1 - 1)^2 - (\lambda/\lambda - 1)^2\), \(\partial E(T)/\partial \lambda_0 < 0\). Since \(\lambda_0 + \lambda_1 > 1\) and \(\lambda_2 > 0\), we have \((-2 + 2\lambda_0 + 2\lambda_1 + \lambda_2) > 0\),
so \(\partial E(T)/\partial \lambda_1 < 0\). Since \(\lambda_0 + \lambda_2 > 1\) and \(\lambda_1 > 0\),
so \(\partial E(T)/\partial \lambda_2 < 0\).
The calculations required for (VII) and (VIII) are as routine as those for (V) and (VI), but they are lot
more lengthy. The proof is complete. \(\square\)

3. Multivariate extension

Consider the following multivariate generalization of (1.1):
\[
F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \left( \frac{x_1}{\beta} \right)^{-\lambda_1} \left( \frac{x_2}{\beta} \right)^{-\lambda_2} \cdots \\
\left( \frac{x_n}{\beta} \right)^{-\lambda_n} \max \left( \frac{x_1}{\beta}, \frac{x_2}{\beta}, \ldots, \frac{x_n}{\beta} \right)^{-\lambda_0},
\]
where \(0 < \beta \leq \min \{x_1, x_2, \ldots, x_n\} < \infty\), and \(\lambda_i > 0\), \(i = 0, 1, \ldots, n\) (Kotz et al. [12, page 595]).
Theorem 2 provides the multivariate analogue of Theorem 1 for \(n(X_1, X_2, \ldots, X_n)\).

Theorem 2. Let \(S = \min(X_1, X_2, \ldots, X_n)\) and \(\lambda = \sum_{i=0}^{n} \lambda_i\). Then
\[
F_S(s) = 1 - \left( \frac{s}{\beta} \right)^{-\lambda},
\]
and
\[
E(S) = \frac{\lambda \beta}{\lambda - 1}, \quad \lambda > 1,
\]
Furthermore, we have the following statements holding:
(I) \(E(S)\) is a monotonically increasing function of \(\beta\) for \(\lambda > 1\),
(II) \(E(S)\) is a monotonically decreasing function of \(\lambda\),
(III) \(Var(S)\) is a monotonically increasing function of \(\beta\) for \(\lambda > 2\),
(IV) \(Var(S)\) is a monotonically decreasing function of \(\lambda\) for \(\lambda > 2\),
(V) \(Var(S)\) is a monotonically decreasing function of \(\lambda\).

Proof: The given expressions for \(F_S(s)\), \(f_S(s)\), \(E(S)\) and \(Var(S)\) are obvious. Since
\[
\frac{\partial E(S)}{\partial \beta} = \frac{\lambda}{\lambda - 1} > 0, \quad \lambda > 1,
\]
statement (I) follows. Since
\[
\frac{\partial E(S)}{\partial \lambda_i} = -\beta \left( \frac{1}{\lambda - 1} \right)^2 < 0, \quad i = 0, 1, \ldots, n,
\]
statement (II) follows. Since
\[
\frac{\partial Var(S)}{\partial \beta} = \frac{2\lambda \beta}{(\lambda - 2)(\lambda - 1)^2} > 0, \quad \lambda > 2,
\]
statement (III) follows. Since
\[
\frac{\partial Var(S)}{\partial \lambda} = -2\beta^2 \left( \frac{\lambda^2 - \lambda - 1}{(\lambda - 2)^2(\lambda - 1)^2} \right) < 0,
\quad \lambda > 2,
\]
statement (IV) follows. A similar analysis shows statement (V). \(\square\)

References

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